Ulam - Hyers Stability of a 2- Variable AC - Mixed Type Functional Equation: Direct and Fixed Point Methods

M. Arunkumar[1], Matina J. Rassias[2], Yanhui Zhang[3]

[1]Department of Mathematics, Government Arts College, Tiruvannamalai - 606 603, TamilNadu, India

[2]Department of Statistical Science, University College London, 1-19 Torrington Place, #140, London, WC1E 7HB, UK

[3]Department of Mathematics, Beijing Technology and Business University, China

[1]annarun2002@yahoo.co.in; [2]matina@stats.ucl.ac.uk; [3]zhangyanhui@th.btbu.edu.cn

***Abstract-* In this paper, we obtain the general solution and generalized Ulam - Hyers stability of a 2 variable AC mixed type functional equation**

$$f\left(2x+y,2z+w\right)-f\left(2x-y,2z-w\right)$$
$$=4\left[f\left(x+y,z+w\right)-f\left(x-y,z-w\right)\right]-6f\left(y,w\right)$$

using direct and fixed point methods.

Keywords-* *Additive Functional Equations; Cubic Functional Equation; Mixed Type AC Functional Equation; Ulam – Hyers Stability

I. INTRODUCTION

The study of stability problems for functional equations is related to a question of Ulam [30] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [9]. It was further generalized and excellent results obtained by number of authors [2], [6], [16], [20], [23].

Over the last six or seven deades, the above problem was tackled by numerous authors and its solutions via various forms of functional equations like additive, quadratic, cubic, quartic, mixed type functional equations which involve only these types of functional equations were discussed. We refer the interested readers for more information on such problems to the monographs [1], [5], [8], [10], [12], [13], [15], [17], [19], [21], [24], [25], [26], [27], [28], [29], [31], [32] and [33].

In 2006, K.W. Jun and H.M. Kim [11] introduced the following generalized additive and quadratic type of functional equation

$$f\left(\sum_{i=1}^{n}x_i\right)+\left(n-2\right)\sum_{i=1}^{n}f\left(x_i\right)=\sum_{1\le i<j\le n}f\left(x_i+x_j\right) \tag{1.1}$$

in the class of function between real vector spaces. The general solution and Ulam stability of mixed type additive and cubic functional equation of the form

$$3f(x+y+z)+f(-x+y+z)+f(x-y+z)$$
$$+f(x+y-z)+4[f(x)+f(y)+f(z)]$$
$$=4[f(x+y)+f(x+z)+f(y+z)] \tag{1.2}$$

was introduced by J.M. Rassias [18]. The stability of generalized mixed type functional equation of the form

$$f(x+ky)+f(x-ky)$$
$$=k^2[f(x+y)+f(x-y)]+2(1-k^2)f(x) \tag{1.3}$$

for fixed integers k and $k\neq 0,\pm 1$ in quasi-Banach spaces was introduced by M. Eshaghi Gordji and H. Khodaie [7]. The mixed type (I.3) is having the property additive, quadratic and cubic.

J.H. Bae and W.G. Park proved the general solution of the 2- variable quadratic functional equation

$$f(x+y,\ z+w)+f(x-y,\ z-w)$$
$$=2f(x,\ z)+2f(y,w) \tag{1.4}$$

and investigated the generalized Hyers-Ulam-Rassias stability of (I.4). The above functional equation has solution

$$f(x,y)=ax^2+bxy+cy^2 \tag{1.5}$$

The stability of the functional equation (I.4) in fuzzy normed space was investigated by M. Arunkumar et., al [3]. Using the ideas in [3], the general solution and generalized Hyers-Ulam- Rassias stability of a 3- variable quadratic functional equation

$$f(x+y,\ z+w,u+v)+f(x-y,z-w,u-v)$$
$$=2f(x,\ z,u)+2f(y,w,v) \tag{1.6}$$

was discussed by K. Ravi and M. Arunkumar [22]. The solution of the (I.6) is of the form

$$f(x,y,z)=ax^2+by^2+cz^2+dxy+eyz+fzx \tag{1.7}$$

In this paper, we obtain the general solution and generalized Ulam - Hyers stability of a 2 variable AC mixed type functional equation

$$f\left(2x+y,2z+w\right)-f\left(2x-y,2z-w\right)$$
$$=4\left[f\left(x+y,z+w\right)-f\left(x-y,z-w\right)\right]-6f\left(y,w\right) \tag{1.8}$$

having solutions

$$f(x,y)=ax+by \tag{1.9}$$

and

$$f(x,y)=ax^3+bx^2y+cxy^2+dy^3 \tag{1.10}$$

In Section 2, we present the general solution of the (I.8). The generalized Ulam-Hyers stability using direct and fixed point method are discussed in Section 3 and Section 4, respectively.

II. GENERAL SOLUTION

In this section, we present the solution of the (I.8). Through out this section let U and V be real vector spaces.

Lemma 2.1: If $f:U^2 \to V$ is a mapping satisfying (I.8) and let $g:U^2 \to V$ be a mapping given by

$$g(x,x) = f(2x,2x) - 8f(x,x) \quad \text{(II.1)}$$

for all $x \in U$ then

$$g(2x,2x) = 2g(x,x) \quad \text{(II.2)}$$

for all $x \in U$ such that g is additive.

Proof: Letting (x,y,z,w) by $(0,0,0,0)$ in (I.8), we get

$$f(0,0) = 0. \quad \text{(II.3)}$$

Setting (x,y,z,w) by $(0,y,0,y)$ in (I.8), we obtain

$$f(-y,-y) = -f(y,y) \quad \text{(II.4)}$$

for all $y \in U$. Replacing (x,y,z,w) by (x,x,x,x) in (I.8), we arrive

$$f(3x,3x) = 4f(2x,2x) - 5f(x,x) \quad \text{(II.5)}$$

for all $x \in U$. Again replacing (x,y,z,w) by $(x,2x,x,2x)$ in (I.8) and using (II.3), (II.4) and (II.5), we have

$$f(4x,4x) = 10f(3x,3x) - 16f(x,x) \quad \text{(II.6)}$$

for all $x \in U$. From (II.1), we establish

$$g(2x,2x) - 2g(x,x) = f(4x,4x) - 10f(2x,2x) + 16f(x,x) \quad \text{(II.7)}$$

for all $x \in U$. Using (II.6) in (II.7), we desired our result.

Lemma 2.2: If $f:U^2 \to V$ be a mapping satisfying (I.8) and let $h:U^2 \to V$ be a mapping given by

$$h(x,x) = f(2x,2x) - 2f(x,x) \quad \text{(II.8)}$$

for all $x \in U$ then

$$h(2x,2x) = 8h(x,x) \quad \text{(II.9)}$$

for all $x \in U$ such that h is cubic.

Proof: It follows from (II.8) that

$$h(2x,2x) - 8g(x,x) = f(4x,4x) - 10f(2x,2x) + 16f(x,x) \quad \text{(II.10)}$$

for all $x \in U$. Using (II.6) in (II.10), we desired our result.

Remark 2.3: If $f:U^2 \to V$ be a mapping satisfying (I.8) and let $g,h:U^2 \to V$ be a mapping defined in (II.1) and (II.8) then

$$f(x,x) = \frac{1}{6}\left(h(x,x) - g(x,x)\right) \quad \text{(II.11)}$$

for all $x \in U$.

Lemma 2.4: If $f:U^2 \to V$ is a mapping satisfying (I.8) and let $t:U \to V$ be a mapping given by

$$t(x) = f(x,x) \quad \text{(II.12)}$$

for all $x \in U$, then t satisfies

$$t(2x+y) - t(2x-y) = 4[t(x+y) - t(x-y)] - 6t(y) \quad \text{(II.13)}$$

for all $x,y \in U$.

Proof: From (I.8) and (II.12), we get

$$\begin{aligned} &t(2x+y) - t(2x-y) \\ &= f(2x+y,2x+y) - f(2x-y,2x-y) \\ &= 4[f(x+y,x+y) - f(x-y,x-y)] - 6f(y,y) \\ &= 4[t(x+y) - t(x-y)] - 6t(y) \end{aligned}$$

for all $x,y \in U$.

III. STABILITY RESULTS: DIRECT METHOD

In this section, we investigate the generalized Ulam-Hyers stability of (I.8) using direct method.

Through out this section let U be a normed space and V be a Banach space. Define a mapping $F:U^2 \to V$ by

$$\begin{aligned} F(x,y,z,w) &= f\left(2x+y,2z+w\right) - f\left(2x-y,2z-w\right) \\ &\quad - 4\left[f\left(x+y,z+w\right) - f\left(x-y,z-w\right)\right] + 6f\left(y,w\right) \end{aligned}$$

for all $x,y,z,w \in U$.

Theorem 3.1: Let $j = \pm 1$. Let $f:U^2 \to V$ be a mapping for which there exist a function $\alpha:U^4 \to (0,\infty]$ with the condition

$$\lim_{n\to\infty} \frac{1}{2^{nj}} \alpha(2^{nj}x, 2^{nj}y, 2^{nj}z, 2^{nj}w) = 0 \quad \text{(III.1)}$$

such that the functional inequality

$$\|F(x,y,z,w)\| \le \alpha(x,y,z,w) \quad \text{(III.2)}$$

for all $x,y,z,w \in U$. Then there exists a unique 2-variable additive mapping $A:U^2 \to V$ satisfying (I.8) and

$$\|f(2x,2x) - 8f(x,x) - A(x,x)\| \le \frac{1}{2} \sum_{k=\frac{1-j}{2}}^{\infty} \beta(2^{kj}x) \quad \text{(III.3)}$$

for all $x \in U$. The mapping $\beta(2^{kj}x)$ and $A(x,x)$ are defined by

$$\begin{aligned} \beta(2^{kj}x) &= 4\alpha(2^{kj}x, 2^{kj}x, 2^{kj}x, 2^{kj}x) + \alpha(2^{kj}x, 2^{(k+1)j}x, 2^{kj}x, 2^{(k+1)j}x) \\ A(x,x) &= \lim_{n\to\infty} \frac{1}{2^{nj}}\left(f(2^{(n+1)j}x, 2^{(n+1)j}x) - 8f(2^{nj}x, 2^{nj}x)\right) \end{aligned} \quad \text{(III.4)}$$

for all $x \in U$.

Proof: Assume $j = 1$. Letting (x,y,z,w) by (x,x,x,x) in (III.2), we obtain

$$\|f(3x,3x) - 4f(2x,2x) + 5f(x,x)\| \le \alpha(x,x,x,x) \quad \text{(III.5)}$$

for all $x \in U$. Replacing (x,y,z,w) by $(x,2x,x,2x)$ in (III.2), we get

$$\|f(4x,4x)-4f(3x,3x)+6f(2x,2x)-4f(x,x)\| \leq \alpha(x,2x,x,2x) \qquad \text{(III.6)}$$

for all $x \in U$. Now, from (III.6) and (III.7), we have

$$\begin{aligned}&\|f(4x,4x)-10f(2x,2x)+16f(x,x)\| \\ &\leq 4\ \|f(3x,3x)-4f(2x,2x)+5f(x,x)\| \\ &\quad +\|f(4x,4x)-4f(3x,3x)+6f(2x,2x)-4f(x,x)\| \\ &\leq 4\alpha(x,x,x,x)+\alpha(x,2x,x,2x) \qquad \text{(III.7)}\end{aligned}$$

for all $x \in U$. From (III.8), we arrive

$$\|f(4x,4x)-10f(2x,2x)+16f(x,x)\| \leq \beta(x) \qquad \text{(III.8)}$$

where

$$\beta(x)=4\alpha(x,x,x,x)+\alpha(x,2x,x,2x) \qquad \text{(III.9)}$$

for all $x \in U$. It is easy to see from (III.9) that

$$\|f(4x,4x)-8f(2x,2x)-2(f(2x,2x)-8f(x;\ x))\| \leq \beta(x) \quad \text{(III.10)}$$

for all $x \in U$. Using (II.1) in (III.11), we obtain

$$\|g(2x,2x)-2g(x,x)\| \leq \beta(x) \qquad \text{(III.11)}$$

for all $x \in U$. From (III.12), we arrive

$$\left\|\frac{g(2x,2x)}{2}-g(x,x)\right\| \leq \frac{\beta(x)}{2} \qquad \text{(III.12)}$$

for all $x \in U$. Now replacing x by $2x$ and dividing by 2 in (III.13), we get

$$\left\|\frac{g(2^2x,2^2x)}{2^2}-\frac{g(x,x)}{2}\right\| \leq \frac{\beta(2x)}{2^2} \qquad \text{(III.13)}$$

for all $x \in U$. From (III.13) and (III.14), we obtain

$$\begin{aligned}&\left\|\frac{g(2^2x,2^2x)}{2^2}-g(x,x)\right\| \\ &\leq \left\|\frac{g(2x,2x)}{2}-g(x,x)\right\|+\left\|\frac{g(2^2x,2^2x)}{2^2}-\frac{g(2x,2x)}{2}\right\| \\ &\leq \frac{1}{2}\left[\beta(x)+\frac{\beta(2x)}{2}\right] \qquad \text{(III.14)}\end{aligned}$$

for all $x \in U$. Proceeding further and using induction on a positive integer n, we get

$$\begin{aligned}\left\|\frac{g(2^nx,2^nx)}{2^n}-g(x,x)\right\| &\leq \frac{1}{2}\sum_{k=0}^{n-1}\frac{\beta(2^kx)}{2} \qquad \text{(III.15)} \\ &\leq \frac{1}{2}\sum_{k=0}^{\infty}\frac{\beta(2^kx)}{2}\end{aligned}$$

for all $x \in U$. In order to prove the convergence of the sequence

$$\left\{\frac{g(2^nx,2^nx)}{2^n}\right\},$$

replacing x by $2^m x$ and dividing by 2^m in (III.16), for any $m, n > 0$, we deduce

$$\begin{aligned}&\left\|\frac{g(2^{n+m}x,2^{n+m}x)}{2^{n+m}}-\frac{g(2^mx,2^mx)}{2^m}\right\| \\ &\leq \frac{1}{2^m}\left\|\frac{g(2^n\cdot 2^mx,2^n\cdot 2^mx)}{2^n}-g(2^mx,2^mx)\right\| \\ &\leq \frac{1}{2}\sum_{k=0}^{\infty}\frac{\beta(2^kx)}{2^{k+m}} \to 0 \quad as \quad m\to\infty\end{aligned}$$

for all $x \in U$. This shows that the sequence $\left\{\frac{g(2^nx,2^nx)}{2^n}\right\}$ is Cauchy sequence. Since V is complete, there exists a mapping $A(x,x):U^2 \to V$ such that

$$A(x,x)=\lim_{n\to\infty}\frac{g(2^nx,2^nx)}{2^n} \quad \forall \quad x\in U.$$

Letting $n\to\infty$ in (III.16) and using (II.1), we see that (III.3) holds for all $x \in U$. To show that A satisfies (I.8), replacing (x,y,z,w) by $(2^nx,2^ny,2^nz,2^nw)$ and dividing by *2n* in (III.2), we obtain

$$\frac{1}{2^n}\|F(2^nx,2^ny,2^nz,2^nw)\| \leq \frac{1}{2^n}\alpha(2^nx,2^ny,2^nz,2^nw)$$

for all $x,y,z,w \in U$. Letting $n\to\infty$ in the above inequality and using the definition of $A(x,x)$, we see that

$$\begin{aligned}&A(2x+y,2z+w)-A(2x-y,2z-w) \\ &=4\left[A(x+y,z+w)-A(x-y,z-w)\right]-6A(y,w)\end{aligned}$$

Hence A satisfies (I.8) for all $x,y,z,w \in U$. To prove A is unique 2-variable additive function satisfying (I.8), we let *B(x,x)* be another 2-variable additive mapping satisfying (I.8) and (III.3), then

$$\begin{aligned}&\|A(x,x)-B(x,x)\| \\ &=\frac{1}{2^n}\|A(2^nx,2^nx)-B(2^nx,2^nx)\| \\ &\leq \frac{1}{2^n}\Big\{\|A(2^nx,2^nx)-f(2^{n+1}x,2^{n+1}x)+8f(2^nx,2^nx)\| \\ &\qquad +\|f(2^{n+1}x,2^{n+1}x)-8f(2^nx,2^nx)-B(2^nx,2^nx)\|\Big\} \\ &\leq \sum_{k=0}^{\infty}\frac{\beta(2^{k+n}x)}{2^{k+n}} \\ &\to 0 \ \ as \ \ n\to\infty\end{aligned}$$

for all $n\to\infty$. Hence A is unique.

For $j=-1$, we can prove a similar stability result. This completes the proof of the theorem.

The following Corollary is an immediate consequence of Theorem 3.1 concerning the stability of (I.8).

Corollary 3.2: Let $F:U^2\to V$ be a mapping and there exists real numbers λ and s such that

$$\|F(x,y,z,w)\| \leq \begin{cases}\lambda \\ \lambda\left\{\|x\|^s+\|y\|^s+\|z\|^s+\|w\|^s\right\}, \ s<1 \ or \ s>1; \\ \lambda\|x\|^s\|y\|^s\|z\|^s\|w\|^s, \ s<\dfrac{1}{4} \ or \ s>\dfrac{1}{4}; \\ \lambda\left\{\|x\|^s\|y\|^s\|z\|^s\|w\|^s+\left\{\|x\|^{4s}+\|y\|^{4s}+\|z\|^{4s}+\|w\|^{4s}\right\}\right\}, \\ \qquad s<\dfrac{1}{4} \ or \ s>\dfrac{1}{4};\end{cases} \qquad \text{(III.17)}$$

for all $x,y,z,w \in U$, then there exists a unique 2- variable additive function $A:U^2\to V$ such that

$$\left\| f(2x,2x)-8f(x,x)-A(x,x)\right\| \le \begin{cases} 5\lambda \\ \dfrac{\left(18+2^{s+1}\right)\lambda\|x\|^{s}}{\left|2-2^{s}\right|} \\ \dfrac{\left(4+2^{2s}\right)\lambda\|x\|^{4s}}{\left|2-2^{4s}\right|} \\ \left(\dfrac{\left(22+2^{2s}\right)}{\left|2-2^{4s}\right|}+\dfrac{2\cdot 2^{4s}}{\left|2-2^{2s}\right|}\right)\lambda\|x\|^{4s} \end{cases} \quad \text{(III.18)}$$

for all $x \in U$.

Now we will provide an example to illustrate that (I.8) is not stable for *s = 1* in (ii) of Corollary 3.2.

Example 3.3: Let $\alpha : R \to R$ be a function defined by

$$\alpha(x) = \begin{cases} \mu x, & if \ |x|<1 \\ \mu, & otherwise \end{cases}$$

where $\mu > 0$ is a constant, and define a function $f : R^2 \to R$ by

$$f(x,x) = \sum_{n=0}^{\infty} \frac{\alpha(2^n x)}{2^n} \qquad for\ all\ x \in R.$$

Then *F* satisfies the functional inequality

$$\left|F(x,y,z,w)\right| \le 32\,\mu\,(|x|+|y|+|z|+|w|) \qquad \text{(III.19)}$$

for all $x,y,z,w \in R$. Then there do not exist a additive mapping $A : R^2 \to R$ and a constant $\beta > 0$ such that

$$\left| f(2x,2x)-8f(x,x)-A(x,x)\right| \le \beta|x| \qquad for\ all\ x \in R. \quad \text{(III.20)}$$

Proof: Now

$$\left|f(x,x)\right| \le \sum_{n=0}^{\infty} \frac{\left|\alpha(2^n x)\right|}{\left|2^n\right|} = \sum_{n=0}^{\infty} \frac{\mu}{\left|2^n\right|} = 2\mu$$

Therefore we see that f is bounded. We are going to prove that f satisfies (III.19).

If *x = y = z = w = 0* then (III.19) is trivial. If $|x|+|y|+|z|+|w| \ge \frac{1}{2}$, then the left hand side of (III.19) is less than 32μ. Now suppose that $0 < |x|+|y|+|z|+|w| < \frac{1}{2}$. Then there exists a positive integer k such that

$$\frac{1}{2^k} \le |x|+|y|+|z|+|w| < \frac{1}{2^{k-1}} \qquad \text{(III.21)}$$

so that

$$2^{k-1}x < \frac{1}{2}, 2^{k-1}y < \frac{1}{2}, 2^{k-1}z < \frac{1}{2}, 2^{k-1}w < \frac{1}{2}$$

and consequently

$$2^{k-1}(y,w), 2^{k-1}(x+y,z+w), 2^{k-1}(x-y,\ z-w),$$
$$2^{k-1}(2x+y,2z+w), 2^{k-1}(2x-y,2z-w), \in (-1,1).$$

Therefore for each $n = 0,1,........k-1$, we have

$$2^{n}(y,w), 2^{n}(x+y,z+w), 2^{n}(x-y,\ z-w),$$
$$2^{n}(2x+y,2z+w), 2^{n}(2x-y,2z-w), \in (-1,1).$$

and

$$\alpha(2^n(2x+y,2z+w))+\alpha(2^n(2x-y,2z-w))$$
$$-4[\alpha(2^n(x+y,z+w))-\alpha(2^n(x-y,\ z-w))]-6\alpha(2^n(y,w))=0$$

For $n = 0,1,........k-1$. From the definition of *f* and (III.21), we obtain that

$$\begin{aligned} &F(x,y,z,w) \\ &\le \sum_{n=0}^{\infty} \frac{1}{2^n}\Big|\alpha(2^n(2x+y,2z+w))+\alpha(2^n(2x-y,2z-w)) \\ &\quad -4[\alpha(2^n(x+y,z+w))-\alpha(2^n(x-y,\ z-w))]-6\alpha(2^n(y,w))\Big| \\ &\le \sum_{n=k}^{\infty} \frac{1}{2^n}\Big|\alpha(2^n(2x+y,2z+w))+\alpha(2^n(2x-y,2z-w)) \\ &\quad -4[\alpha(2^n(x+y,z+w))-\alpha(2^n(x-y,\ z-w))]-6\alpha(2^n(y,w))\Big| \\ &\le \sum_{n=k}^{\infty} \frac{1}{2^n}16\mu = 16\mu\frac{2}{2^k} = 32\,\mu\,(|x|+|y|+|z|+|w|) \end{aligned}$$

Thus *f* satisfies (III.19) for all $x,y,z,w \in R$ with

$$0 < |x|+|y|+|z|+|w| < \frac{1}{2}$$

We claim that the additive (I.8) is not stable for *s = 1* in (ii) Corollary 3.2. Suppose on the contrary that there exist a additive mapping $A : R^2 \to R$ and a constant $\beta > 0$ satisfying (III.20). Since *f* is bounded and continuous for all $x \in R$, *A* is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 3.1. A must have the form $A(x,x) = cx$ for any x in R. Thus we obtain that

$$\left|f(2x,2x)-8f(x,x)\right| \le (\beta+|c|)\ |x|, \qquad \text{(III.22)}$$

But we can choose a positive integer *m* with $m\beta > \beta + |c|$.

If $x \in \left(0, \frac{1}{2^{m-1}}\right)$, then $2^n x \in (0,1)$ for all $n = 0, 1,.....m-1$.

For this *x*, we get

$$f(2x,2x)-8f(x,x) = \sum_{n=0}^{\infty} \frac{\alpha(2^n x)}{2^n} \ge \sum_{n=0}^{m-1} \frac{\mu(2^n x)}{2^n} = m\mu x > (\beta+|c|)\,x$$

which contradicts (III.22). Therefore (I.8) is not stable in sense of Ulam, Hyers and Rassias if *s = 1*, assumed in (ii) of (III.17).

A counter example to illustrate the non stability in (iii) of Corollary 3.2 is given in the following example.

Example 3.4: Let *s* be such that $0 < s < \frac{1}{4}$. Then there is a function $F : R^2 \to R$ and a constant $\lambda > 0$ satisfying

$$\left|F(x,y,z,w)\right| \le \lambda\,|x|^{\frac{s}{4}}|y|^{\frac{s}{4}}|z|^{\frac{s}{4}}|w|^{\frac{1-3s}{4}} \qquad \text{(III.23)}$$

for all $x,y,z,w \in R$ and

$$\sup_{x\neq 0} \frac{\left|f(2x,2x)-8f(x,x)-A(x,x)\right|}{|x|} = +\infty \qquad \text{(III.24)}$$

for every additive mapping $A(x,x): R^2 \to R$.

Proof: If we take

$$f\ (x,x) = \begin{cases} (x,x)\ln(x,x), & if\ x \neq 0 \\ 0, & if\ x = 0 \end{cases}$$

Then from (III.24), it follows that

$$\sup_{x\neq 0} \frac{\left| f(2x,2x) - 8f(x,x) - A(x,x) \right|}{|x|}$$
$$\geq \sup_{\substack{n\in\mathbb{N} \\ n\neq 0}} \frac{\left| f(2n,2n) - 8f(n,n) - A(n,n) \right|}{|n|}$$
$$= \sup_{\substack{n\in\mathbb{N} \\ n\neq 0}} \frac{\left| n(2,2)\ln|2n,2n| - 8n(1,1)\ln|n,n| - nA(1,1) \right|}{|n|}$$
$$= \sup_{\substack{n\in\mathbb{N} \\ n\neq 0}} \left| (2,2)\ln|2n,2n| - 8(1,1)\ln|n,n| - A(1,1) \right| = +\infty$$

We have to prove (III.23) is true.

Case (i): If $x,y,z,w > 0$ in (III.23) then,

$$\left| f(2x+y,2z+w) - f(2x-y,2z-w) - 4f(x+y,z+w) \right.$$
$$\left. +4f(x-y,z-w) + 6f(y,w) \right|$$
$$= \left| (2x+y,2z+w)\ln|2x+y,2z+w| \right.$$
$$-(2x-y,2z-w)\ln|2x-y,2z-w|$$
$$-4(x+y,z+w)\ln|x+y,z+w|$$
$$\left. +4(x-y,z-w)\ln|x-y,z-w| + 6(y,w)\ln|y,w| \right|$$

Set $x = v_1, y = v_2, z = v_3, w = v_4$ it follows that

$$\left| f(2x+y,2z+w) - f(2x-y,2z-w) - 4f(x+y,z+w) \right.$$
$$\left. +4f(x-y,z-w) + 6f(y,w) \right|$$
$$= \left| (2v_1+v_2,2v_3+v_4)\ln|2v_1+v_2,2v_3+v_4| \right.$$
$$-(2v_1-v_2,2v_3-v_4)\ln|2v_1-v_2,2v_3-v_4|$$
$$-4(v_1+v_2,v_3+v_4)\ln|v_1+v_2,v_3+v_4|$$
$$\left. +4(v_1-v_2,v_3-v_4)\ln|v_1-v_2,v_3-v_4| + 6(v_2,v_4)\ln|v_2,v_4| \right|$$
$$= \left| f(2v_1+v_2,2v_3+v_4) - f(2v_1-v_2,2v_3-v_4) \right.$$
$$-4f(v_1+v_2,v_3+v_4)$$
$$\left. +4f(v_1-v_2,v_3-v_4) + 6f(v_2,v_4) \right|$$
$$\leq \lambda |v_1|^{\frac{s}{4}} |v_2|^{\frac{s}{4}} |v_3|^{\frac{s}{4}} |v_4|^{\frac{1-3s}{4}}$$
$$= \lambda |x|^{\frac{s}{4}} |y|^{\frac{s}{4}} |z|^{\frac{s}{4}} |w|^{\frac{1-3s}{4}}$$

Case (ii): If $x,y,z,w < 0$ in (III.23) then,

$$\left| f(2x+y,2z+w) - f(2x-y,2z-w) - 4f(x+y,z+w) \right.$$
$$\left. +4f(x-y,z-w) + 6f(y,w) \right|$$
$$= \left| (2x+y,2z+w)\ln|2x+y,2z+w| \right.$$
$$-(2x-y,2z-w)\ln|2x-y,2z-w|$$
$$-4(x+y,z+w)\ln|x+y,z+w|$$
$$\left. +4(x-y,z-w)\ln|x-y,z-w| + 6(y,w)\ln|y,w| \right|$$

Set $-x = v_1, -y = v_2, -z = v_3, -w = v_4$ it follows that

$$\left| f(2x+y,2z+w) - f(2x-y,2z-w) - 4f(x+y,z+w) \right.$$
$$\left. +4f(x-y,z-w) + 6f(y,w) \right|$$
$$= \left| (-2v_1-v_2,-2v_3-v_4)\ln|-2v_1-v_2,-2v_3-v_4| \right.$$
$$-(-2v_1+v_2,-2v_3+v_4)\ln|-2v_1+v_2,-2v_3+v_4|$$
$$-4(-v_1-v_2,-v_3-v_4)\ln|-v_1-v_2,-v_3-v_4|$$
$$\left. +4(-v_1+v_2,-v_3+v_4)\ln|-v_1+v_2,-v_3+v_4| + 6(-v_2,-v_4)\ln|-v_2,-v_4| \right|$$
$$= \left| f(-2v_1-v_2,-2v_3-v_4) - f(-2v_1+v_2,-2v_3+v_4) \right.$$
$$-4f(-v_1-v_2,-v_3-v_4)$$
$$\left. +4f(-v_1+v_2,-v_3+v_4) + 6f(-v_2,-v_4) \right|$$
$$\leq \lambda |-v_1|^{\frac{s}{4}} |-v_2|^{\frac{s}{4}} |-v_3|^{\frac{s}{4}} |-v_4|^{\frac{1-3s}{4}}$$
$$= \lambda |x|^{\frac{s}{4}} |y|^{\frac{s}{4}} |z|^{\frac{s}{4}} |w|^{\frac{1-3s}{4}}$$

Case (iii): If $x,z > 0,\ y,w < 0$ then $2x+y$, $2z+w,\ x+y, z+w > 0, 2x-y, 2z-w, x-y,\ z-w < 0$ in (III.23) then,

$$\left| f(2x+y,2z+w) - f(2x-y,2z-w) - 4f(x+y,z+w) \right.$$
$$\left. +4f(x-y,z-w) + 6f(y,w) \right|$$
$$= \left| (2x+y,2z+w)\ln|2x+y,2z+w| \right.$$
$$-(2x-y,2z-w)\ln|2x-y,2z-w|$$
$$-4(x+y,z+w)\ln|x+y,z+w|$$
$$\left. +4(x-y,z-w)\ln|x-y,z-w| + 6(y,w)\ln|y,w| \right|$$

Set $x = v_1, -y = v_2, z = v_3, -w = v_4$ it follows that

$$\left| f(2x+y,2z+w) - f(2x-y,2z-w) - 4f(x+y,z+w) \right.$$
$$\left. +4f(x-y,z-w) + 6f(y,w) \right|$$
$$= \left| (2v_1-v_2,2v_3-v_4)\ln|2v_1-v_2,2v_3-v_4| \right.$$
$$-(2v_1+v_2,2v_3+v_4)\ln|-(2v_1+v_2,2v_3+v_4)|$$
$$-4(v_1-v_2,v_3-v_4)\ln|v_1-v_2,v_3-v_4|$$
$$\left. +4(v_1+v_2,v_3+v_4)\ln|-(v_1+v_2,v_3+v_4)| + 6(-v_2,-v_4)\ln|-v_2,-v_4| \right|$$
$$= \left| f(2v_1-v_2,2v_3-v_4) - f(2v_1+v_2,2v_3+v_4) \right.$$
$$-4f(v_1-v_2,v_3-v_4)$$
$$\left. +4f(v_1+v_2,v_3+v_4) + 6f(-v_2,-v_4) \right|$$
$$\leq \lambda |v_1|^{\frac{s}{4}} |-v_2|^{\frac{s}{4}} |v_3|^{\frac{s}{4}} |-v_4|^{\frac{1-3s}{4}}$$
$$= \lambda |x|^{\frac{s}{4}} |y|^{\frac{s}{4}} |z|^{\frac{s}{4}} |w|^{\frac{1-3s}{4}}$$

Case (iv): If $x,z > 0,\ y,w < 0$ then $2x+y$, $2z+w,\ x+y, z+w < 0, 2x-y, 2z-w, x-y,\ z-w > 0$ in (III.23) then,

$$\left| f(2x+y,2z+w) - f(2x-y,2z-w) - 4f(x+y,z+w) \right.$$
$$\left. +4f(x-y,z-w) + 6f(y,w) \right|$$
$$= \left| (2x+y,2z+w)\ln|2x+y,2z+w| \right.$$
$$-(2x-y,2z-w)\ln|2x-y,2z-w|$$
$$-4(x+y,z+w)\ln|x+y,z+w|$$
$$\left. +4(x-y,z-w)\ln|x-y,z-w| + 6(y,w)\ln|y,w| \right|$$

Set $x = v_1, -y = v_2, z = v_3, -w = v_4$ it follows that

$$\begin{aligned}
&|f(2x+y,2z+w)-f(2x-y,2z-w)-4f(x+y,z+w)\\
&+4f(x-y,z-w)+6f(y,w)|\\
&=|(2v_1-v_2,2v_3-v_4)\ln|-(2v_1-v_2,2v_3-v_4)|\\
&-(2v_1+v_2,2v_3+v_4)\ln|2v_1+v_2,2v_3+v_4|\\
&-4(v_1-v_2,v_3-v_4)\ln|-(v_1-v_2,v_3-v_4)|\\
&+4(v_1+v_2,v_3+v_4)\ln|v_1+v_2,v_3+v_4|+6(-v_2,-v_4)\ln|-v_2,-v_4||\\
&=|f(2v_1-v_2,2v_3-v_4)-f(2v_1+v_2,2v_3+v_4)\\
&-4f(v_1-v_2,v_3-v_4)\\
&+4f(v_1+v_2,v_3+v_4)+6f(-v_2,-v_4)|\\
&\le\lambda|v_1|^{\frac{s}{4}}|-v_2|^{\frac{s}{4}}|v_3|^{\frac{s}{4}}|-v_4|^{\frac{1-3s}{4}}\\
&=\lambda|x|^{\frac{s}{4}}|y|^{\frac{s}{4}}|z|^{\frac{s}{4}}|w|^{\frac{1-3s}{4}}
\end{aligned}$$

Case (v): If $x=y=z=w=0$ in (III.23) then it is trivial.

Now we will provide an example to illustrate that the (I.8) is not stable for $s=\frac{1}{4}$ in (iv) of Corollary 3.2.

Example 3.5: Let $\alpha:R\to R$ be a function defined by

$$\alpha(x)=\begin{cases}\mu x, & if\ |x|<\frac{1}{4}\\ \frac{\mu}{4}, & otherwise\end{cases}$$

where $\mu>0$ is a constant, and define a function $f:E^2\to R$ by

$$f(x,x)=\sum_{n=0}^{\infty}\frac{\alpha(2^n x)}{2^n}\qquad for\ all\ x\in R.$$

Then *F* satisfies the functional inequality

$$|F(x,y,z,w)|$$
$$\le 8\mu\left(|x|^{\frac14}|y|^{\frac14}|z|^{\frac14}|w|^{\frac14}+\{|x|+|y|+|z|+|w|\}\right)\quad\text{(III.25)}$$

for all $x,y,z,w\in R$. Then there do not exist a additive mapping $A:R^2\to R$ and a constant $\beta>0$ such that

$$|f(2x,2x)-8f(x,x)-A(x,x)|\le\beta|x|\qquad for\ all\ x\in R.\ \text{(III.26)}$$

Proof: Now

$$|f(x,x)|\le\sum_{n=0}^{\infty}\frac{|\alpha(2^n x)|}{|2^n|}=\sum_{n=0}^{\infty}\frac{1}{|2^n|}\times\frac{\mu}{4}=\frac{\mu}{2}.$$

Therefore we see that *f* is bounded. We are going to prove that f satisfies (III.25).

If *x = y = z = w = 0* then (III.25) is trivial. If $|x|^{\frac14}|y|^{\frac14}|z|^{\frac14}|w|^{\frac14}+\{|x|+|y|+|z|+|w|\}\ge\frac{1}{2}$, then the left hand side of (III.25) is less than 8μ. Now suppose that

$$0<|x|^{\frac14}|y|^{\frac14}|z|^{\frac14}|w|^{\frac14}+\{|x|+|y|+|z|+|w|\}<\frac{1}{2}.$$

Then there exists a positive integer k such that

$$\frac{1}{2^k}\le|x|^{\frac14}|y|^{\frac14}|z|^{\frac14}|w|^{\frac14}+\{|x|+|y|+|z|+|w|\}<\frac{1}{2^{k-1}}\quad\text{(III.27)}$$

so that

$$2^{k-1}|x|^{\frac14}|y|^{\frac14}|z|^{\frac14}|w|^{\frac14}<\frac12,\ 2^{k-1}x<\frac12,\ 2^{k-1}y<\frac12,$$
$$2^{k-1}z<\frac12,\ 2^{k-1}w<\frac12$$

and consequently

$$2^{k-1}(y,w),2^{k-1}(x+y,z+w),2^{k-1}(x-y,\ z-w),$$
$$2^{k-1}(2x+y,2z+w),2^{k-1}(2x-y,2z-w),\in\left(-\frac14,\frac14\right).$$

Therefore for each $n=0,1,\ldots\ldots,k-1$, we have

$$2^n(y,w),2^n(x+y,z+w),2^n(x-y,\ z-w),$$
$$2^n(2x+y,2z+w),2^n(2x-y,2z-w)\in\left(-\frac14,\frac14\right).$$

and

$$\alpha(2^n(2x+y,2z+w))+\alpha(2^n(2x-y,2z-w))$$
$$-4[\alpha(2^n(x+y,z+w))-\alpha(2^n(x-y,\ z-w))]-6\alpha(2^n(y,w))=0$$

For $n=0,1,\ldots\ldots,k-1$. From the definition of *f* and (III.27), we obtain that

$$\begin{aligned}
&F(x,y,z,w)\\
&\le\sum_{n=0}^{\infty}\frac{1}{2^n}\Big|\alpha(2^n(2x+y,2z+w))+\alpha(2^n(2x-y,2z-w))\\
&-4[\alpha(2^n(x+y,z+w))-\alpha(2^n(x-y,\ z-w))]-6\alpha(2^n(y,w))\Big|\\
&\le\sum_{n=k}^{\infty}\frac{1}{2^n}\Big|\alpha(2^n(2x+y,2z+w))+\alpha(2^n(2x-y,2z-w))\\
&-4[\alpha(2^n(x+y,z+w))-\alpha(2^n(x-y,\ z-w))]-6\alpha(2^n(y,w))\Big|\\
&\le\sum_{n=k}^{\infty}\frac{1}{2^n}\frac{16\mu}{4}=4\mu\frac{2}{2^k}\\
&=8\mu\left(|x|^{\frac14}|y|^{\frac14}|z|^{\frac14}|w|^{\frac14}+\{|x|+|y|+|z|+|w|\}\right)
\end{aligned}$$

Thus *f* satisfies (III.25) for all $x,y,z,w\in$ with

$$0<|x|^{\frac14}|y|^{\frac14}|z|^{\frac14}|w|^{\frac14}+\{|x|+|y|+|z|+|w|\}<\frac12$$

We claim that (I.8) is not stable for $s=\frac14$ in (iii) Corollary 3.2. Suppose on the contrary that there exist a additive mapping $A:R^2\to R$ and a constant $\beta>0$ satisfying (III.26). Since *f* is bounded and continuous for all $x\in R$, *A* is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 3.1, A must have the form $A(x,x)=cx$ for any x in R. Thus we obtain that

$$|f(2x,2x)-8f(x,x)|\le(\beta+|c|)\ |x|,\quad\text{(III.28)}$$

But we can choose a positive integer *m* with $m\beta>\beta+|c|$.

If $x\in\left(0,\frac{1}{2^{m-1}}\right)$, then $2^n x\in(0,1)$ for all $n=0,\ 1,\ldots..m-1$. For this *x*, we get

$$f(2x,2x)-8f(x,x)=\sum_{n=0}^{\infty}\frac{\alpha(2^n x)}{2^n}\geq\sum_{n=0}^{m-1}\frac{\mu(2^n x)}{2^n}$$
$$=m\mu x>(\beta+|c|)\,x$$

which contradicts (III.28). Therefore (I.8) is not stable in sense of Ulam, Hyers and Rassias if $s=\frac{1}{4}$, assumed in (iv) of (III.17).

Theorem 3.6: Let $j=\pm1$. Let $f:U^2\to V$ be a mapping for which there exist a function $\alpha:U^4\to(0,\infty]$ with the condition

$$\lim_{n\to\infty}\frac{1}{8^{nj}}\alpha(2^{nj}x,2^{nj}y,2^{nj}z,2^{nj}w)=0 \tag{III.29}$$

such that the functional inequality

$$\|F(x,y,z,w)\|\leq\alpha(x,y,z,w) \tag{III.30}$$

for all $x,y,z,w\in U$. Then there exists a unique 2-variable cubic mapping $C:U^2\to V$ satisfying (I.8) and

$$\|f(2x,2x)-2f(x,x)-C(x,x)\|\leq\frac{1}{8}\sum_{k=\frac{1-j}{2}}^{\infty}\beta(2^{kj}x) \tag{III.31}$$

for all $x\in U$. The mapping $\beta(2^{kj}x)$ and $C(x,x)$ are defined by

$$\beta(2^{kj}x)=4\alpha(2^{kj}x,2^{kj}x,2^{kj}x,2^{kj}x)+\alpha(2^{kj}x,2^{(k+1)j}x,2^{kj}x,2^{(k+1)j}x)$$
$$C(x,x)=\lim_{n\to\infty}\frac{1}{8^{nj}}\left(f(2^{(n+1)j}x,2^{(n+1)j}x)-2f(2^{nj}x,2^{nj}x)\right) \tag{III.32}$$

for all $x\in U$.

Proof: It is easy to see from (III.9) that

$$\|f(4x,4x)-2f(2x,2x)-8(f(2x,2x)-2f(x;\,x))\|\leq\beta(x) \tag{III.34}$$

for all $x\in U$. Using (II.8) in (III.34), we obtain

$$\|h(2x,2x)-8h(x,x)\|\leq\beta(x) \tag{III.35}$$

for all $x\in U$. From (III.35), we arrive

$$\left\|\frac{h(2x,2x)}{8}-h(x,x)\right\|\leq\frac{\beta(x)}{8} \tag{III.36}$$

for all $x\in U$. Now replacing x by $2x$ and dividing by 8 in (III.36), we get

$$\left\|\frac{h(2^2x,2^2x)}{8^2}-\frac{h(x,x)}{8}\right\|\leq\frac{\beta(2x)}{8^2} \tag{III.37}$$

for all $x\in U$. From (III.36) and (III.37), we obtain

$$\left\|\frac{h(2^2x,2^2x)}{8^2}-h(x,x)\right\|$$
$$\leq\left\|\frac{h(2x,2x)}{8}-h(x,x)\right\|+\left\|\frac{h(2^2x,2^2x)}{8^2}-\frac{h(2x,2x)}{8}\right\|$$
$$\leq\frac{1}{8}\left[\beta(x)+\frac{\beta(2x)}{8}\right] \tag{III.38}$$

for all $x\in U$. Proceeding further and using induction on a positive integer n, we get

$$\left\|\frac{h(2^nx,2^nx)}{8^n}-h(x,x)\right\|\leq\frac{1}{8}\sum_{k=0}^{n-1}\frac{\beta(2^kx)}{8^k}$$
$$\leq\frac{1}{8}\sum_{k=0}^{\infty}\frac{\beta(2^kx)}{8^k} \tag{III.39}$$

for all $x\in U$. In order to prove the convergence of the sequence

$$\left\{\frac{h(2^nx,2^nx)}{8^n}\right\},$$

replacing x by 2^mx and dividing by 8^m in (III.39), for any *m, n > 0*, we deduce

$$\left\|\frac{h(2^{n+m}x,2^{n+m}x)}{8^{n+m}}-\frac{h(2^mx,2^mx)}{8^m}\right\|$$
$$\leq\frac{1}{8^m}\left\|\frac{h(2^n\cdot2^mx,2^n\cdot2^mx)}{8^n}-h(2^mx,2^mx)\right\|$$
$$\leq\frac{1}{8}\sum_{k=0}^{\infty}\frac{\beta(2^kx)}{8^{k+m}}$$
$$\to\ 0\quad as\quad m\to\infty$$

for all $x\in U$. This shows that the sequence $\left\{\frac{h(2^nx,2^nx)}{8^n}\right\}$ is Cauchy sequence. Since V is complete, there exists a mapping $C(x,x):U^2\to V$ such that

$$C(x,x)=\lim_{n\to\infty}\frac{h(2^nx,2^nx)}{2^n}\quad\forall\quad x\in U.$$

Letting $n\to\infty$ in (III.39) and using (II.8), we see that (III.31) holds for all $x\in U$. To show that C satisfies (I.8) and it is unique the proof is similar to that of Theorem 3.1

The following Corollary is an immediate consequence of Theorem 3.6 concerning the stability of (I.8).

Corollary 3.7: Let $F:U^2\to V$ be a mapping and there exists real numbers λ and s such that

$$\|F(x,y,z,w)\|$$
$$\leq\begin{cases}\lambda\\ \lambda\left\{\|x\|^s+\|y\|^s+\|z\|^s+\|w\|^s\right\},\ s<3\ or\ s>3;\\ \lambda\|x\|^s\|y\|^s\|z\|^s\|w\|^s,\ s<\dfrac{3}{4}\ or\ s>\dfrac{3}{4};\\ \lambda\left\{\|x\|^s\|y\|^s\|z\|^s\|w\|^s\right.\\ \qquad\left.\left\{\|x\|^{4s}+\|y\|^{4s}+\|z\|^{4s}+\|w\|^{4s}\right\}\right\},\\ \qquad s<\dfrac{3}{4}\ or\ s>\dfrac{3}{4};\end{cases} \tag{III.40}$$

for all $x,y,z,w\in U$, then there exists a unique 2- variable cubic function $C:U^2\to V$ such that

$$\|f(2x,2x)-2f(x,x)-C(x,x)\|$$

$$\le \begin{cases} 5\lambda/7 \\ \dfrac{\left(18+2^{s+1}\right)\lambda\|x\|^{s}}{7\left|8-2^{s}\right|} \\ \dfrac{\left(4+2^{2s}\right)\lambda\|x\|^{4s}}{7\left|8-2^{4s}\right|} \\ \left(\dfrac{\left(22+2^{2s}\right)}{7\left|8-2^{4s}\right|}+\dfrac{2\cdot 2^{4s}}{7\left|8-2^{2s}\right|}\right)\lambda\|x\|^{4s} \end{cases} \qquad \text{(III.41)}$$

for all $x \in U$.

Now we will provide an example to illustrate that (I.8) is not stable for $s = 3$ in (ii) of Corollary 3.6.

Example 3.8: Let $\alpha : R \to R$ be a function defined by

$$\alpha(x) = \begin{cases} \mu x^{3}, & if \ |x|<1 \\ \mu, & otherwise \end{cases}$$

where $\mu > 0$ is a constant, and define a function $f : R^2 \to R$ by

$$f(x,x) = \sum_{n=0}^{\infty} \frac{\alpha(2^n x)}{8^n} \qquad for\ all\ x \in R.$$

Then F satisfies the functional inequality

$$\left|F(x,y,z,w)\right| \le \frac{16\mu \times 8^3}{7}(|x|^3+|y|^3+|z|^3+|w|^3) \qquad \text{(III.42)}$$

for all $x,y,z,w \in R$. Then there do not exist a cubic mapping $C : R^2 \to R$ and a constant $\beta > 0$ such that

$$\left|f(2x,2x) - 2f(x,x) - C(x,x)\right| \le \beta |x|^3 \quad for\ all\ x \in R. \qquad \text{(III.43)}$$

Proof: Now

$$\left|f(x,x)\right| \le \sum_{n=0}^{\infty} \frac{\left|\alpha(2^n x)\right|}{\left|8^n\right|} = \sum_{n=0}^{\infty} \frac{\mu}{\left|8^n\right|} = \frac{8\mu}{7}$$

Therefore we see that f is bounded. We are going to prove that f satisfies (III.42).

If $x = y = z = w = 0$, then (III.42) is trivial. If $|x|^3+|y|^3+|z|^3+|w|^3 \ge \frac{1}{8}$, then the left hand side of (III.42) is less than $\frac{16\mu\times 8}{7}$. Now suppose that

$$0 < |x|^3+|y|^3+|z|^3+|w|^3 < \frac{1}{8}.$$

Then there exists a positive integer k such that

$$\frac{1}{8^{k+2}} \le |x|^3+|y|^3+|z|^3+|w|^3 < \frac{1}{8^{k+1}} \qquad \text{(III.44)}$$

so that

$$2^{k-1}x^3 < \frac{1}{8}, 2^{k-1}y^3 < \frac{1}{8}, 2^{k-1}z^3 < \frac{1}{8}, 2^{k-1}w^3 < \frac{1}{8}$$

and consequently

$$2^{k-1}(y,w), 2^{k-1}(x+y,z+w), 2^{k-1}(x-y,\ z-w),$$
$$2^{k-1}(2x+y,2z+w), 2^{k-1}(2x-y,2z-w) \in \left(-\frac{1}{2},\frac{1}{2}\right).$$

Therefore for each $n = 0,1,\ldots\ldots,k-1$, we have

$$2^{n}(y,w), 2^{n}(x+y,z+w), 2^{n}(x-y,\ z-w),$$
$$2^{n}(2x+y,2z+w), 2^{n}(2x-y,2z-w) \in \left(-\frac{1}{2},\frac{1}{2}\right).$$

and

$$\alpha(2^n(2x+y,2z+w)) + \alpha(2^n(2x-y,2z-w))$$
$$-4[\alpha(2^n(x+y,z+w)) - \alpha(2^n(x-y,\ z-w))] - 6\alpha(2^n(y,w)) = 0$$

For $n = 0,1,\ldots\ldots,k-1$. From the definition of f and (III.44), we obtain that

$$\begin{aligned} &F(x,y,z,w) \\ &\le \sum_{n=0}^{\infty} \frac{1}{2^n}\left|\alpha(2^n(2x+y,2z+w)) + \alpha(2^n(2x-y,2z-w))\right. \\ &\qquad \left. -4[\alpha(2^n(x+y,z+w)) - \alpha(2^n(x-y,\ z-w))] - 6\alpha(2^n(y,w))\right| \\ &\le \sum_{n=k}^{\infty} \frac{1}{2^n}\left|\alpha(2^n(2x+y,2z+w)) + \alpha(2^n(2x-y,2z-w))\right. \\ &\qquad \left. -4[\alpha(2^n(x+y,z+w)) - \alpha(2^n(x-y,\ z-w))] - 6\alpha(2^n(y,w))\right| \\ &\le \sum_{n=k}^{\infty} \frac{1}{8^n} 16\mu = \frac{16\mu\times 8}{7} \times \frac{1}{8^k} \\ &= \frac{16\mu \times 8^3}{7}(|x|^3+|y|^3+|z|^3+|w|^3) \end{aligned}$$

Thus f satisfies (III.42) for all $x,y,z,w \in$ with

$$0 < |x|^3+|y|^3+|z|^3+|w|^3 < \frac{1}{8}$$

We claim that (I.8) is not stable for $s = 3$ in (ii) Corollary 3.7. Suppose on the contrary that there exist a cubic mapping $C : R^2 \to R$ and a constant $\beta > 0$ satisfying (III.43). Since f is bounded and continuous for all $x \in R$, C is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 3.6, C must have the form $C(x,x) = cx^3$ for any x in R. Thus we obtain that

$$\left|f(2x,2x) - 2f(x,x)\right| \le (\beta + |c|)\ |x|^3, \qquad \text{(III.45)}$$

But we can choose a positive integer m with $m\beta > \beta + |c|$.

If $x \in \left(0, \frac{1}{2^{m-1}}\right)$, then $2^n x \in (0,1)$ for all $n = 0,\ 1,\ldots..m-1$. For this x, we get

$$f(2x,2x) - 2f(x,x) = \sum_{n=0}^{\infty} \frac{\alpha(2^n x)}{8^n} \ge \sum_{n=0}^{m-1} \frac{\mu(2^n x)^3}{8^n}$$
$$= m\mu x^3 > (\beta + |c|)\, x^3$$

which contradicts (III.45). Therefore (I.8) is not stable in sense of Ulam, Hyers and Rassias if $s = 3$, assumed in the inequality (ii) of (III.41).

A counter example to illustrate the non stability in (iii) of Corollary 3.7 is given in the following example.

Example 3.9: Let s be such that $0 < s < \frac{3}{4}$. Then there is a function $F : R^2 \to R$ and a constant $\lambda > 0$ satisfying

$$\left|F(x,y,z,w)\right| \le \lambda |x|^{\frac{s}{4}} |y|^{\frac{s}{4}} |z|^{\frac{s}{4}} |w|^{\frac{3-3s}{4}} \qquad \text{(III.46)}$$

for all $x,y,z,w \in R$ and

$$\sup_{x\neq 0}\frac{|f(2x,2x)-2f(x,x)-C(x,x)|}{|x|^3}=+\infty \qquad \text{(III.47)}$$

for every cubic mapping $C(x,x): R^2 \to R$.

Proof: If we take

$$f\ (x,x)=\begin{cases}(x,x)^3\ln(x,x), & if\ \ x\neq 0\\ 0, & if\ \ x=0\end{cases}$$

Then from (III.47), it follows that

$$\begin{aligned}
&\sup_{x\neq 0}\frac{|f(2x,2x)-2f(x,x)-C(x,x)|}{|x|^3}\\
&\geq \sup_{\substack{n\in\mathbb{N}\\ n\neq 0}}\frac{|f(2n,2n)-2f(n,n)-C(n,n)|}{|n|^3}\\
&=\sup_{\substack{n\in\mathbb{N}\\ n\neq 0}}\frac{|n^3(2,2)^3\ln|2n,2n|-2n^3(1,1)^3\ln|n,n|-n^3C(1,1)|}{|n|^3}\\
&=\sup_{\substack{n\in\mathbb{N}\\ n\neq 0}}|(2,2)^3\ln|2n,2n|-8(1,1)^3\ln|n,n|-C(1,1)|=+\infty
\end{aligned}$$

We have to prove (III.46) is true.

Case (i): If $x,y,z,w>0$ in (III.46) then,

$$\begin{aligned}
&|f(2x+y,2z+w)-f(2x-y,2z-w)-4f(x+y,z+w)\\
&+4f(x-y,z-w)+6f(y,w)|\\
&=|(2x+y,2z+w)^3\ln|2x+y,2z+w|\\
&\quad-(2x-y,2z-w)^3\ln|2x-y,2z-w|\\
&\quad-4(x+y,z+w)^3\ln|x+y,z+w|\\
&\quad+4(x-y,z-w)^3\ln|x-y,z-w|+6(y,w)^3\ln|y,w||
\end{aligned}$$

Set $x=v_1, y=v_2, z=v_3, w=v_4$ it follows that

$$\begin{aligned}
&|f(2x+y,2z+w)-f(2x-y,2z-w)-4f(x+y,z+w)\\
&+4f(x-y,z-w)+6f(y,w)|\\
&=|(2v_1+v_2,2v_3+v_4)^3\ln|2v_1+v_2,2v_3+v_4|\\
&\quad-(2v_1-v_2,2v_3-v_4)^3\ln|2v_1-v_2,2v_3-v_4|\\
&\quad-4(v_1+v_2,v_3+v_4)^3\ln|v_1+v_2,v_3+v_4|\\
&\quad+4(v_1-v_2,v_3-v_4)^3\ln|v_1-v_2,v_3-v_4|+6(v_2,v_4)^3\ln|v_2,v_4||\\
&=|f(2v_1+v_2,2v_3+v_4)-f(2v_1-v_2,2v_3-v_4)\\
&\quad-4f(v_1+v_2,v_3+v_4)\\
&\quad+4f(v_1-v_2,v_3-v_4)+6f(v_2,v_4)|\\
&\leq \lambda|v_1|^{\frac{s}{4}}|v_2|^{\frac{s}{4}}|v_3|^{\frac{s}{4}}|v_4|^{\frac{1-3s}{4}}\\
&=\lambda|x|^{\frac{s}{4}}|y|^{\frac{s}{4}}|z|^{\frac{s}{4}}|w|^{\frac{1-3s}{4}}
\end{aligned}$$

Case (ii): If $x,y,z,w<0$ in (III.46) then,

$$\begin{aligned}
&|f(2x+y,2z+w)-f(2x-y,2z-w)-4f(x+y,z+w)\\
&+4f(x-y,z-w)+6f(y,w)|\\
&=|(2x+y,2z+w)^3\ln|2x+y,2z+w|\\
&\quad-(2x-y,2z-w)^3\ln|2x-y,2z-w|\\
&\quad-4(x+y,z+w)^3\ln|x+y,z+w|\\
&\quad+4(x-y,z-w)^3\ln|x-y,z-w|+6(y,w)^3\ln|y,w||
\end{aligned}$$

Set $-x=v_1,-y=v_2,-z=v_3,-w=v_4$ it follows that

$$\begin{aligned}
&|f(2x+y,2z+w)-f(2x-y,2z-w)-4f(x+y,z+w)\\
&+4f(x-y,z-w)+6f(y,w)|\\
&=|(-2v_1-v_2,-2v_3-v_4)^3\ln|-2v_1-v_2,-2v_3-v_4|\\
&\quad-(-2v_1+v_2,-2v_3+v_4)^3\ln|-2v_1+v_2,-2v_3+v_4|\\
&\quad-4(-v_1-v_2,-v_3-v_4)^3\ln|-v_1-v_2,-v_3-v_4|\\
&\quad+4(-v_1+v_2,-v_3+v_4)^3\ln|-v_1+v_2,-v_3+v_4|+6(-v_2,-v_4)^3\ln|-v_2,-v_4||\\
&=|f(-2v_1-v_2,-2v_3-v_4)-f(-2v_1+v_2,-2v_3+v_4)\\
&\quad-4f(-v_1-v_2,-v_3-v_4)\\
&\quad+4f(-v_1+v_2,-v_3+v_4)+6f(-v_2,-v_4)|\\
&\leq \lambda|-v_1|^{\frac{s}{4}}|-v_2|^{\frac{s}{4}}|-v_3|^{\frac{s}{4}}|-v_4|^{\frac{1-3s}{4}}\\
&=\lambda|x|^{\frac{s}{4}}|y|^{\frac{s}{4}}|z|^{\frac{s}{4}}|w|^{\frac{1-3s}{4}}
\end{aligned}$$

Case (iii): If $x,z>0,\ y,w<0$ then $2x+y,$ $2z+w,\ x+y,z+w>0, 2x-y,2z-w,x-y,\ z-w<0$ in (III.46) then,

$$\begin{aligned}
&|f(2x+y,2z+w)-f(2x-y,2z-w)-4f(x+y,z+w)\\
&+4f(x-y,z-w)+6f(y,w)|\\
&=|(2x+y,2z+w)^3\ln|2x+y,2z+w|\\
&\quad-(2x-y,2z-w)^3\ln|2x-y,2z-w|\\
&\quad-4(x+y,z+w)^3\ln|x+y,z+w|\\
&\quad+4(x-y,z-w)^3\ln|x-y,z-w|+6(y,w)^3\ln|y,w||
\end{aligned}$$

Set $x=v_1,-y=v_2,z=v_3,-w=v_4$ it follows that

$$\begin{aligned}
&|f(2x+y,2z+w)-f(2x-y,2z-w)-4f(x+y,z+w)\\
&+4f(x-y,z-w)+6f(y,w)|\\
&=|(2v_1-v_2,2v_3-v_4)^3\ln|2v_1-v_2,2v_3-v_4|\\
&\quad-(2v_1+v_2,2v_3+v_4)^3\ln|-(2v_1+v_2,2v_3+v_4)|\\
&\quad-4(v_1-v_2,v_3-v_4)^3\ln|v_1-v_2,v_3-v_4|\\
&\quad+4(v_1+v_2,v_3+v_4)^3\ln|-(v_1+v_2,v_3+v_4)|+6(-v_2,-v_4)^3\ln|-v_2,-v_4||\\
&=|f(2v_1-v_2,2v_3-v_4)-f(2v_1+v_2,2v_3+v_4)\\
&\quad-4f(v_1-v_2,v_3-v_4)\\
&\quad+4f(v_1+v_2,v_3+v_4)+6f(-v_2,-v_4)|\\
&\leq \lambda|v_1|^{\frac{s}{4}}|-v_2|^{\frac{s}{4}}|v_3|^{\frac{s}{4}}|-v_4|^{\frac{1-3s}{4}}\\
&=\lambda|x|^{\frac{s}{4}}|y|^{\frac{s}{4}}|z|^{\frac{s}{4}}|w|^{\frac{1-3s}{4}}
\end{aligned}$$

Case (iv): If $x,z>0,\ y,w<0$ then $2x+y,$ $2z+w,\ x+y,z+w<0, 2x-y,2z-w,x-y,\ z-w>0$ in (III.46) then,

$$\begin{aligned}
&|f(2x+y,2z+w)-f(2x-y,2z-w)-4f(x+y,z+w)\\
&+4f(x-y,z-w)+6f(y,w)|\\
&=|(2x+y,2z+w)^3\ln|2x+y,2z+w|\\
&\quad-(2x-y,2z-w)^3\ln|2x-y,2z-w|\\
&\quad-4(x+y,z+w)^3\ln|x+y,z+w|\\
&\quad+4(x-y,z-w)^3\ln|x-y,z-w|+6(y,w)^3\ln|y,w||
\end{aligned}$$

Set $x=v_1,-y=v_2,z=v_3,-w=v_4$ it follows that

$$
\begin{aligned}
&\left| f(2x+y,2z+w) - f(2x-y,2z-w) - 4f(x+y,z+w) \right.\\
&\left. +4f(x-y,z-w) + 6f(y,w) \right|\\
&= \left| (2v_1-v_2,2v_3-v_4)^3 \ln|-(2v_1-v_2,2v_3-v_4)| \right.\\
&\quad -(2v_1+v_2,2v_3+v_4)^3 \ln|2v_1+v_2,2v_3+v_4|\\
&\quad -4(v_1-v_2,v_3-v_4)^3 \ln|-(v_1-v_2,v_3-v_4)|\\
&\quad \left. +4(v_1+v_2,v_3+v_4)^3 \ln|v_1+v_2,v_3+v_4| + 6(-v_2,-v_4)^3 \ln|-v_2,-v_4| \right|\\
&= \left| f(2v_1-v_2,2v_3-v_4) - f(2v_1+v_2,2v_3+v_4) \right.\\
&\quad -4f(v_1-v_2,v_3-v_4)\\
&\quad \left. +4f(v_1+v_2,v_3+v_4) + 6f(-v_2,-v_4) \right|\\
&\le \lambda |v_1|^{\frac{s}{4}} |-v_2|^{\frac{s}{4}} |v_3|^{\frac{s}{4}} |-v_4|^{\frac{1-3s}{4}}\\
&= \lambda |x|^{\frac{s}{4}} |y|^{\frac{s}{4}} |z|^{\frac{s}{4}} |w|^{\frac{1-3s}{4}}
\end{aligned}
$$

Case (v): If $x=y=z=w=0$ in (III.46) then it is trivial.

Now we will provide an example to illustrate that (I.8) is not stable for $s=\frac{3}{4}$ in (iv) of Corollary 3.7.

Example 3.10: Let $\alpha: R \to R$ be a function defined by

$$
\alpha(x) = \begin{cases} \mu x^3, & if \ |x| < \frac{3}{4} \\ \frac{3\mu}{4}, & otherwise \end{cases}
$$

where $\mu > 0$ is a constant, and define a function $f: R^2 \to R$ by

$$
f(x,x) = \sum_{n=0}^{\infty} \frac{\alpha(2^n x)}{8^n} \qquad for\ all\ x \in R.
$$

Then F satisfies the functional inequality

$$
\left| F(x,y,z,w) \right| \le \frac{96\mu \times 8^2}{7} \left(|x|^{\frac{3}{4}} |y|^{\frac{3}{4}} |z|^{\frac{3}{4}} |w|^{\frac{3}{4}} + \left\{ |x|^3 + |y|^3 + |z|^3 + |w|^3 \right\} \right) \tag{III.48}
$$

for all $x,y,z,w \in R$. Then there do not exist a cubic mapping $C: R^2 \to R$ and a constant $\beta > 0$ such that

$$
\left| f(2x,2x) - 2f(x,x) - C(x,x) \right| \le \beta |x|^3 \qquad for\ all\ x \in R. \tag{III.49}
$$

Proof: Now

$$
\left| f(x,x) \right| \le \sum_{n=0}^{\infty} \frac{\left| \alpha(2^n x) \right|}{\left| 8^n \right|} = \sum_{n=0}^{\infty} \frac{1}{\left| 8^n \right|} \times \frac{3\mu}{4} = \frac{6\mu}{7}.
$$

Therefore we see that f is bounded. We are going to prove that f satisfies (III.48).

If $x = y = z = w = 0$, then (III.48) is trivial. If $|x|^{\frac{3}{4}} |y|^{\frac{3}{4}} |z|^{\frac{3}{4}} |w|^{\frac{3}{4}} + \left\{ |x|^3 + |y|^3 + |z|^3 + |w|^3 \right\} \ge \frac{1}{8}$, then the left hand side of (III.48) is less than $\frac{96\mu}{7}$. Now suppose that

$$
0 < |x|^{\frac{3}{4}} |y|^{\frac{3}{4}} |z|^{\frac{3}{4}} |w|^{\frac{3}{4}} + \left\{ |x|^3 + |y|^3 + |z|^3 + |w|^3 \right\} < \frac{1}{8}.
$$

Then there exists a positive integer k such that

$$
\frac{1}{8^{k+2}} \le |x|^{\frac{3}{4}} |y|^{\frac{3}{4}} |z|^{\frac{3}{4}} |w|^{\frac{3}{4}} + \left\{ |x|^3 + |y|^3 + |z|^3 + |w|^3 \right\} < \frac{1}{8^{k+1}} \tag{III.50}
$$

so that

$$
2^{k-1} |x|^{\frac{1}{4}} |y|^{\frac{1}{4}} |z|^{\frac{1}{4}} |w|^{\frac{1}{4}} < \frac{1}{2}, 2^{k-1} x < \frac{1}{2}, 2^{k-1} y < \frac{1}{2}, \quad 2^{k-1} z < \frac{1}{2}, 2^{k-1} w < \frac{1}{2}
$$

and consequently

$$
\begin{aligned}
&2^{k-1}(y,w), 2^{k-1}(x+y,z+w), 2^{k-1}(x-y,\ z-w),\\
&2^{k-1}(2x+y,2z+w), 2^{k-1}(2x-y,2z-w), \in \left(-\frac{1}{2}, \frac{1}{2} \right).
\end{aligned}
$$

Therefore for each $n = 0,1,\ldots\ldots,k-1$, we have

$$
\begin{aligned}
&2^n(y,w), 2^n(x+y,z+w), 2^n(x-y,\ z-w),\\
&2^n(2x+y,2z+w), 2^n(2x-y,2z-w) \in \left(-\frac{1}{2}, \frac{1}{2} \right).
\end{aligned}
$$

and

$$
\begin{aligned}
&\alpha(2^n(2x+y,2z+w)) + \alpha(2^n(2x-y,2z-w))\\
&-4[\alpha(2^n(x+y,z+w)) - \alpha(2^n(x-y,\ z-w))] - 6\alpha(2^n(y,w)) = 0
\end{aligned}
$$

For $n = 0,1,\ldots\ldots,k-1$. From the definition of f and (III.50), we obtain that

$$
\begin{aligned}
&F(x,y,z,w)\\
&\le \sum_{n=0}^{\infty} \frac{1}{8^n} \left| \alpha(2^n(2x+y,2z+w)) + \alpha(2^n(2x-y,2z-w)) \right.\\
&\quad \left. -4[\alpha(2^n(x+y,z+w)) - \alpha(2^n(x-y,\ z-w))] - 6\alpha(2^n(y,w)) \right|\\
&\le \sum_{n=k}^{\infty} \frac{1}{8^n} \left| \alpha(2^n(2x+y,2z+w)) + \alpha(2^n(2x-y,2z-w)) \right.\\
&\quad \left. -4[\alpha(2^n(x+y,z+w)) - \alpha(2^n(x-y,\ z-w))] - 6\alpha(2^n(y,w)) \right|\\
&\le \sum_{n=k}^{\infty} \frac{1}{8^n} \frac{16\mu \times 3}{4} = \frac{12\mu \times 8}{7} \frac{1}{8^k}\\
&= 8\,\mu \left(|x|^{\frac{3}{4}} |y|^{\frac{3}{4}} |z|^{\frac{3}{4}} |w|^{\frac{3}{4}} + \left\{ |x|^3 + |y|^3 + |z|^3 + |w|^3 \right\} \right)
\end{aligned}
$$

Thus f satisfies (III.48) for all $x,y,z,w \in$ □ with

$$
0 < |x|^{\frac{3}{4}} |y|^{\frac{3}{4}} |z|^{\frac{3}{4}} |w|^{\frac{3}{4}} + \left\{ |x|^3 + |y|^3 + |z|^3 + |w|^3 \right\} < \frac{1}{8}
$$

We claim that (I.8) is not stable for $s=\frac{3}{4}$ in (iii) Corollary 3.7. Suppose on the contrary that there exist a cubic mapping $C: R^2 \to R$ and a constant $\beta > 0$ satisfying (III.49). Since f is bounded and continuous for all $x \in R$, C is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 3.6, C must have the form $C(x,x) = cx^3$ for any x in R. Thus we obtain that

$$
\left| f(2x,2x) - 2f(x,x) \right| \le (\beta + |c|)\ |x|^3, \tag{III.51}
$$

But we can choose a positive integer m with $m\beta > \beta + |c|$.

If $x \in \left(0, \frac{1}{2^{m-1}}\right)$, then $2^n x \in (0,1)$ for all $n = 0, 1, \ldots, m-1$. For this x, we get

$$\begin{aligned} f(2x,2x) - 2f(x,x) &= \sum_{n=0}^{\infty} \frac{\alpha(2^n x)}{8^n} \geq \sum_{n=0}^{m-1} \frac{\mu(2^n x)^3}{8^n} \\ &= m\mu x^3 > (\beta + |c|)\, x^3 \end{aligned}$$

which contradicts (III.51). Therefore (I.8) is not stable in sense of Ulam, Hyers and Rassias if $s = \frac{3}{4}$, assumed in (iv) of (III.41).

Now, we are ready to prove our main stability results.

Theorem 3.11: Let $j = \pm 1$. Let $f : U^2 \to V$ be a mapping for which there exist a function $\alpha : U^4 \to (0, \infty]$ with the condition given in (III.1) and (III.29) respectively, such that the functional inequality

$$\|F(x,y,z,w)\| \leq \alpha(x,y,z,w) \tag{III.52}$$

for all $x,y,z,w \in U$. Then there exists a unique 2-variable additive mapping $A : U^2 \to V$ unique 2-variable cubic mapping $C : U^2 \to V$ satisfying (I.8) and

$$\|f(x,x) - A(x,x) - C(x,x)\| \leq \frac{1}{6}\left\{ \frac{1}{2} \sum_{k=\frac{1-j}{2}}^{\infty} \beta(2^{kj}x) + \frac{1}{8} \sum_{k=\frac{1-j}{2}}^{\infty} \beta(2^{kj}x) \right\} \tag{III.53}$$

for all $x \in U$. The mapping $\beta(2^{kj}x)$, $A(x,x)$ and $C(x,x)$ are defined in (III.4), (III.5) and (III.33)

for all $x \in U$.

Proof: By Theorems 3.1 and 3.6, there exists a unique 2-variable additive function $A_1 : U^2 \to V$ and a unique 2-variable cubic function $C_1 : U^2 \to V$ such that

$$\|f(2x,2x) - 8f(x,x) - A_1(x,x)\| \leq \frac{1}{2} \sum_{k=\frac{1-j}{2}}^{\infty} \beta(2^{kj}x) \tag{III.54}$$

and

$$\|f(2x,2x) - 2f(x,x) - C_1(x,x)\| \leq \frac{1}{8} \sum_{k=\frac{1-j}{2}}^{\infty} \beta(2^{kj}x) \tag{III.55}$$

for all $x \in U$. Now from (III.54) and (III.55), one can see that

$$\begin{aligned} &\left\| f(x,x) + \frac{1}{6}A_1(x,x) - \frac{1}{6}C_1(x,x) \right\| \\ &= \left\| \left\{ -\frac{f(2x,2x)}{6} + \frac{8f(x,x)}{6} + \frac{A_1(x,x)}{6} \right\} + \left\{ \frac{f(2x,2x)}{6} - \frac{2f(x,x)}{6} - \frac{C_1(x,x)}{6} \right\} \right\| \\ &\leq \frac{1}{6}\{\|f(2x,2x) - 8f(x,x) - A_1(x,x)\| + \|f(2x,2x) - 8f(x,x) - A_1(x,x)\|\} \\ &\leq \frac{1}{6}\left\{ \frac{1}{2} \sum_{k=\frac{1-j}{2}}^{\infty} \beta(2^{kj}x) + \frac{1}{8} \sum_{k=\frac{1-j}{2}}^{\infty} \beta(2^{kj}x) \right\} \end{aligned}$$

for all $x \in U$. Thus we obtain (III.55) by defining $A_1(x,x) = \frac{-1}{6}A(x,x)$ and $C_1(x,x) = \frac{1}{6}C(x,x)$, $\beta(2^{kj}x)$, $A(x,x)$ and $C(x,x)$ are defined in (III.4), (III.5) and (III.33)

for all $x \in U$.

The following corollary is the immediate consequence of Theorem 3.11, using Corollaries 3.2 and 3.7 concerning the stability of (I.8).

Corollary 3.12: Let $F : U^2 \to V$ be a mapping and there exists real numbers λ and s such that

$$\|F(x,y,z,w)\| \leq \begin{cases} \lambda \\ \lambda\left\{\|x\|^s + \|y\|^s + \|z\|^s + \|w\|^s\right\}, \; s < 3 \;\; or \;\; s > 3; \\ \lambda \|x\|^s \|y\|^s \|z\|^s \|w\|^s, \; s < \frac{3}{4} \;\; or \;\; s > \frac{3}{4}; \\ \lambda\left\{\|x\|^s \|y\|^s \|z\|^s \|w\|^s + \left\{\|x\|^{4s} + \|y\|^{4s} + \|z\|^{4s} + \|w\|^{4s}\right\}\right\}, \\ \qquad s < \frac{3}{4} \;\; or \;\; s > \frac{3}{4}; \end{cases} \tag{III.56}$$

for all $x,y,z,w \in U$, then there exists a unique 2-variable additive mapping $A : U^2 \to V$ unique 2-variable cubic mapping $C : U^2 \to V$ such that

$$\|f(x,x) - A(x,x) - C(x,x)\| \leq \begin{cases} \frac{5\lambda}{6}\left(1 + \frac{1}{7}\right) \\ \frac{(18 + 2^{s+1})}{6}\left(\frac{1}{|2 - 2^s|} + \frac{1}{7|8 - 2^s|}\right)\lambda\|x\|^s \\ \frac{(4 + 2^{2s})}{6}\left(\frac{1}{|2 - 2^{4s}|} + \frac{1}{7|8 - 2^{4s}|}\right)\lambda\|x\|^{4s} \\ \frac{(22 + 2^{2s})}{6}\left(\frac{1}{|2 - 2^{4s}|} + \frac{1}{7|8 - 2^{2s}|}\right)\lambda\|x\|^{4s} \\ \frac{2 \cdot 2^{4s}}{6}\left(\frac{1}{|2 - 2^{4s}|} + \frac{1}{7|8 - 2^{2s}|}\right)\lambda\|x\|^{4s} \end{cases} \tag{III.57}$$

for all $x \in U$.

Now we will provide an example to illustrate that (I.8) is not stable for $s = 1$ in (ii) of Corollary 3.12.

Example 3.13: Let $\alpha : R \to R$ be a function defined by

$$\alpha(x) = \begin{cases} \mu(x + x^3), & if \;\; |x| < 1 \\ \mu, & otherwise \end{cases}$$

where $\mu > 0$ is a constant, and define a function $f : R^2 \to R$ by

$$f(x,x) = \sum_{n=0}^{\infty} \frac{\alpha(2^n x)}{2^n} \qquad for \; all \; x \in R.$$

Then F satisfies the functional inequality

$$|F(x,y,z,w)| \le 32\mu(|x|+|y|+|z|+|w|) \qquad \text{(III.58)}$$

for all $x,y,z,w \in R$. Then there do not exist a 2-variable additive mapping $A:U^2 \to V$ and 2-variable cubic mapping $C:U^2 \to V$ and a constant $\beta > 0$ such that

$$|f(x,x) - A(x,x) - C(x,x)| \le \beta|x| \quad \text{for all } x \in R. \quad \text{(III.59)}$$

A counter example to illustrate the non stability in (iii) of Corollary 3.12 is given in the following example.

Example 3.14: Let s be such that $0 < s < \frac{1}{4}$. Then there is a function $F: R^2 \to R$ and a constant $\lambda > 0$ satisfying

$$|F(x,y,z,w)| \le \lambda |x|^{\frac{s}{4}} |y|^{\frac{s}{4}} |z|^{\frac{s}{4}} |w|^{\frac{1-3s}{4}} \qquad \text{(III.60)}$$

for all $x,y,z,w \in R$ and

$$\sup_{x \ne 0} \frac{|f(x,x) - A(x,x) - C(x,x)|}{|x|} = +\infty \qquad \text{(III.61)}$$

for every additive mapping $A(x,x): R^2 \to R$, and for every cubic mapping $C(x,x): R^2 \to R$.

Now we will provide an example to illustrate that (I.8) is not stable for $s = \frac{1}{4}$ in (iv) of Corollary 3.12.

Example 3.15: Let $\alpha: R \to R$ be a function defined by

$$\alpha(x) = \begin{cases} \mu(x + x^3), & if \quad |x| < \frac{1}{4} \\ \frac{\mu}{4}, & otherwise \end{cases}$$

where $\mu > 0$ is a constant, and define a function $f: R^2 \to R$ by

$$f(x,x) = \sum_{n=0}^{\infty} \frac{\alpha(2^n x)}{2^n} \qquad \text{for all } x \in R.$$

Then F satisfies the functional inequality

$$|F(x,y,z,w)| \le 8\mu \left(|x|^{\frac{1}{4}} |y|^{\frac{1}{4}} |z|^{\frac{1}{4}} |w|^{\frac{1}{4}} + \{|x|+|y|+|z|+|w|\} \right) \qquad \text{(III.62)}$$

for all $x,y,z,w \in R$. Then there do not exist a 2-variable additive mapping $A:U^2 \to V$ and a cubic mapping $C: R^2 \to R$ and a constant $\beta > 0$ such that

$$|f(2x,2x) - 2f(x,x) - C(x,x)| \le \beta|x|^3 \qquad \text{for all } x \in R. \text{(III.63)}$$

IV. STABILITY RESULTS:FIXED POINT METHOD

In this section, we apply a fixed point method for achieving stability of the 2-variable AC (I.8).

Now, we present the following theorem due to B. Margolis and J.B. Diaz [14] for fixed point Theory.

Theorem 4.1: Suppose that for a complete generalized metric space (Ω, d) and a strictly contractive mapping $T: \Omega \to \Omega$ with Lipschitz constant L. Then, for each given $x \in \Omega$, either $d(T^n x, T^{n+1}x) = \infty$ for all $n \ge 0$, or there exists a natural number n_0 such that

i) $d(T^n x, T^{n+1}x) < \infty$ for all $n \ge n_0$;

ii) The sequence $(T^n x)$ is convergent to a fixed to a fixed point y^* of T

iii) y^* is the unique fixed point of T in the set

$$\Delta = \{ y \in \Omega : d(T^{n_0} x, y) < \infty \};$$

iv) $d(y, y^*) \le \frac{1}{1-L} d(y, Ty)$ for all $y \in \Delta$.

Using the above theorem, we now obtain the generalized Hyers-Ulam-Rassias stability of (I.8).

Thorough out this section, let U be a vector space and V Banach space. Define a mapping $F: U^2 \to V$ by

$$F(x,y,z,w) = f(2x+y, 2z+w) - f(2x-y, 2z-w) - 4[f(x+y, z+w) - f(x-y, z-w)] + 6f(y,w)$$

for all $x,y,z,w \in U$.

Theorem 4.2: Let $f: U^2 \to V$ be a mapping for which there exists a function $\alpha: U^4 \to (0,\infty]$ with the condition

$$\lim_{n\to\infty} \tfrac{1}{\mu_i^n} \alpha(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n w) = 0 \qquad \text{(IV.1)}$$

with $\mu_i = 2$ if $i = 0$ and $\mu_i = \frac{1}{2}$ if $i = 1$ such that the functional inequality

$$\|F(x,y,z,w)\| \le \alpha(x,y,z,w) \qquad \text{(IV.2)}$$

for all $x,y,z,w \in U$. If there exists $L = L(i) < 1$ such that the function

$$x \to \gamma(x) = \tfrac{1}{2}\beta(x),$$

has the property

$$\gamma(x) \le L \cdot \mu_i \, \gamma(\mu_i x). \qquad \text{(IV.3)}$$

Then there exists a unique 2-variable additive mapping $A: U^2 \to V$ satisfying (I.8) and

$$\|f(2x,2x) - 8f(x,x) - A(x,x)\| \le \frac{L^{i-1}}{1-L} \gamma(x) \qquad \text{(IV.4)}$$

for all $x \in U$. The mapping $\beta(2^{kj} x)$ and $A(x,x)$ are defined in (III.10) and (III.5) respectively for all $x \in U$.

Proof: Consider the set

$$\Omega = \{ p / p: U^2 \to V, p(0) = 0 \}$$

and introduce the generalized metric on Ω,

$$d(g,h) = d_\gamma(g,h) = \inf\{ K \in (0,\infty) / \|p(x) - q(x)\| \le K\gamma(x), x \in U \}$$

It is easy to see that (Ω, d) is complete.

Define $T:\Omega^2 \to \Omega$ by

$$Tp(x,x) = \frac{1}{\mu_i} p(\mu_i x, \mu_i x)$$

for all $x \in U$. Now for all $p,q \in \Omega$,

$$\begin{aligned}
&d(g,h) \le K \\
&\Rightarrow \|p(x,x) - q(x,x)\| \le K\gamma(x),\ x \in U, \\
&\Rightarrow \left\|\frac{1}{\mu_i} p(\mu_i x, \mu_i x) - \frac{1}{\mu_i} q(\mu_i x, \mu_i x)\right\| \le \frac{1}{\mu} K\gamma(\mu_i x),\ x \in U, \\
&\Rightarrow \left\|\frac{1}{\mu_i} p(\mu_i x, \mu_i x) - \frac{1}{\mu_i} q(\mu_i x, \mu_i x)\right\| \le LK\gamma(x),\ x \in U, \\
&\Rightarrow \|T\,p(x,x) - T\,q(x,x)\| \le LK\gamma(x),\ x \in U, \\
&\Rightarrow d_\gamma(Tp,Tq) \le LK.
\end{aligned}$$

This gives $d(Tp,Tq) \le Ld(p,q)$, for all $p,q \in \Omega$, i.e., T is a strictly contractive mapping of Ω, with Lipschitz constant L. From (III.12), we arrive

$$\left\|\frac{g(2x,2x)}{2} - g(x,x)\right\| \le \frac{\beta(x)}{2} \qquad \text{(IV.5)}$$

for all $x \in U$. Using (IV.3) for the case $i = 0$ it reduces to

$$\left\|\frac{g(2x,2x)}{2} - g(x,x)\right\| \le L\gamma(x) \text{ for all } x \in U$$

i.e., $d_\gamma(g,Tg) \le L \Rightarrow d(g,Tg) \le L \le L^1 < \infty$

Again replacing $x = \frac{x}{2}$, in (IV.5), we get,

$$\left\|g(x,x) - 2g\left(\frac{x}{2},\frac{x}{2}\right)\right\| \le \beta\left(\frac{x}{2}\right) \qquad \text{(IV.6)}$$

for all $x \in U$. Using (IV.3) for the case $i = 1$ it reduces to

$$\left\|g(x,x) - 2g\left(\frac{x}{2},\frac{x}{2}\right)\right\| \le \gamma(x) \text{ for all } x \in U,$$

i.e., $d_\gamma(g,Tg) \le 1 \Rightarrow d(g,Tg) \le 1 \le L^0 < \infty$

Now from the fixed point alternative in both cases, it follows that there exists a fixed point A of T in Ω such that

$$A(x,x) = \lim_{n\to\infty} \frac{1}{\mu_i^n}\left(f(\mu_i^{n+1}x, \mu_i^{n+1}x) - 8f(\mu_i^n x, \mu_i^n x)\right) \qquad \text{(IV.7)}$$

for all $x \in U$, since $\lim_{n\to\infty} d(T^n g, A) = 0$.

To prove $A:U^2 \to V$ is additive. Putting (x,y,z,w) by $\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n w$ in (IV.2) and divide by μ_i^n. It follows from (IV.1) that

$$\begin{aligned}
\|A(x,y,z,w)\| &= \lim_{n\to\infty} \frac{\|F(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n w)\|}{\mu_i^n} \\
&\le \lim_{n\to\infty} \frac{\alpha(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n w)}{\mu_i^n} = 0
\end{aligned}$$

for all $x,y,z,w \in U$, i.e., A satisfies (I.8).

According to the alternative fixed point, since A is the unique fixed point of T in the set $\Delta = \{p \in \Omega : d(f,p) < \infty\}, A$ is the unique function such that

$$\|f(2x,2x) - 8f(x,x) - A(x,x)\| \le K\gamma(x)$$

for all $x \in U$ and $K > 0$. Again using the fixed point alternative, we obtain

$$d(f,A) \le \frac{1}{1-L} d(f,Tf)$$

this implies

$$d(f,A) \le \frac{L^{1-i}}{1-L} d(f,Tf)$$

which yields

$$\|f(2x,2x) - 8f(x,x) - A(x,x)\| \le \frac{L^{1-i}}{1-L}\gamma(x)$$

this completes the proof of the theorem.

The following Corollary is an immediate consequence of Theorem 4.2 concerning the stability of (I.8).

Corollary 4.3: Let $F:U^2 \to V$ be a mapping and there exists real numbers λ and s such that

$$\|F(x,y,z,w)\| \le \begin{cases} \lambda\left\{\|x\|^s + \|y\|^s + \|z\|^s + \|w\|^s\right\}, & s < 1 \text{ or } s > 1; \\ \lambda \|x\|^s \|y\|^s \|z\|^s \|w\|^s, & s < \frac{1}{4} \text{ or } s > \frac{1}{4}; \\ \lambda\left\{\|x\|^s \|y\|^s \|z\|^s \|w\|^s + \left\{\|x\|^{4s} + \|y\|^{4s} + \|z\|^{4s} + \|w\|^{4s}\right\}\right\}, & s < \frac{1}{4} \text{ or } s > \frac{1}{4}; \end{cases} \qquad \text{(IV.8)}$$

for all $x,y,z,w \in U$, then there exists a unique 2- variable additive function $A:U^2 \to V$ such that

$$\|f(2x,2x) - 8f(x,x) - A(x,x)\| \le \begin{cases} \dfrac{2^{s-1}(18 + 2^{s+1})\lambda\|x\|^s}{|2 - 2^s|} \\ \dfrac{2^{4s-1}(4 + 2^{2s})\lambda\|x\|^{4s}}{|2 - 2^{4s}|} \\ \dfrac{2^{4s-1}(22 + 2^{2s} + 2\cdot 2^{4s})\lambda\|x\|^{4s}}{|2 - 2^{4s}|} \end{cases} \qquad \text{(IV.9)}$$

for all $x \in U$.

Proof: Setting

$$\alpha(x,y,z) = \begin{cases} \lambda\left\{\|x\|^s + \|y\|^s + \|z\|^s + \|w\|^s\right\}, \\ \lambda \|x\|^s \|y\|^s \|z\|^s \|w\|^s, \\ \lambda\left\{\|x\|^s \|y\|^s \|z\|^s \|w\|^s + \left\{\|x\|^{4s} + \|y\|^{4s} + \|z\|^{4s} + \|w\|^{4s}\right\}\right\}, \end{cases}$$

for all $x, y, z, w \in U$. Then, for $s<2$ if $i=0$ and for $s>1$ if $i=1$ we get

$$\frac{\alpha\left(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n w\right)}{\mu_i^n}$$

$$= \begin{cases} \frac{\lambda}{\mu_i^n}\left\{\left\|\mu_i^n x\right\|^s + \left\|\mu_i^n y\right\|^s + \left\|\mu_i^n z\right\|^s + \left\|\mu_i^n w\right\|^s\right\}, \\ \frac{\lambda}{\mu_i^n}\left\|\mu_i^n x\right\|^s \left\|\mu_i^n y\right\|^s \left\|\mu_i^n z\right\|^s \left\|\mu_i^n w\right\|^s, \\ \frac{\lambda}{\mu_i^n}\left\{\left\|\mu_i^n x\right\|^s \left\|\mu_i^n y\right\|^s \left\|\mu_i^n z\right\|^s \left\|\mu_i^n w\right\|^s \right. \\ \qquad \left.\left\{\left\|\mu_i^n x\right\|^{4s} + \left\|\mu_i^n y\right\|^{4s} + \left\|\mu_i^n z\right\|^{4s} + \left\|\mu_i^n w\right\|^{4s}\right\}\right\}, \end{cases}$$

$$= \begin{cases} \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty, \end{cases}$$

Thus (IV.1) is holds. But we have $x \to \gamma(x) = \frac{1}{2}\beta(x)$, has the property $\gamma(x) \le L \cdot \mu_i\, \gamma\left(\mu_i x\right)$, for all $x \in U$. Hence

$$\gamma(x) = \frac{1}{2}\beta(x)$$
$$= \frac{1}{2}(4\alpha(x,x,x,x) + \alpha(x,2x,x,2x))$$
$$= \begin{cases} \frac{\lambda}{2}(18\,\|x\|^s + 2\,\|2x\|^s) \\ \frac{\lambda}{2}(4+2^{2s})\,\|x\|^{4s} \\ \frac{\lambda}{2}(22+2^{2s}+2\cdot 2^{4s})\,\|x\|^{4s} \end{cases}$$

Now

$$\frac{1}{\mu_i}\gamma\left(\mu_i x\right) = \begin{cases} \frac{\lambda}{2\mu_i}(18\,\|\mu_i x\|^s + 2\,\|2\mu_i x\|^s) \\ \frac{\lambda}{2\mu_i}(4+2^{2s})\,\|\mu_i x\|^{4s} \\ \frac{\lambda}{2\mu_i}(22+2^{2s}+2\cdot 2^{4s})\,\|\mu_i x\|^{4s} \end{cases}$$

$$= \begin{cases} \frac{\lambda}{2\mu_i}\mu_i^s(18\,\|x\|^s + 2\,\|2x\|^s) \\ \frac{\lambda}{2\mu_i}\mu_i^{4s}(4+2^{2s})\,\|x\|^{4s} \\ \frac{\lambda}{2\mu_i}\mu_i^{4s}(22+2^{2s}+2\cdot 2^{4s})\,\|x\|^{4s} \end{cases}$$

$$= \begin{cases} \frac{\lambda}{2}\mu_i^{s-1}(18\,\|x\|^s + 2\,\|2x\|^s) \\ \frac{\lambda}{2}\mu_i^{4s-1}(4+2^{2s})\,\|x\|^{4s} \\ \frac{\lambda}{2}\mu_i^{4s-1}(22+2^{2s}+2\cdot 2^{4s})\,\|x\|^{4s} \end{cases}$$

$$= \begin{cases} \mu_i^{s-1}\beta(x) \\ \mu_i^{4s-1}\beta(x) \\ \mu_i^{4s-1}\beta(x) \end{cases}$$

for all $x \in U$.

Hence the (IV.3) holds either, $L = 2^{s-1}$ for $s<1$ if $i=0$ and $L = \frac{1}{2^{s-1}}$ for $s>1$ if $i=1$.

Now from (IV.4), we prove the following cases for (i).

Case (i): $L = 2^{s-1}$ for $s<1$ if $i=0$.

$$\|f(2x,2x) - 8f(x,x) - A(x,x)\|$$
$$\le \frac{\left(2^{s-1}\right)^{1-0}}{1-2^{s-1}}\left\{\frac{18+2^{s+1}}{2}\right\}\lambda\,\|x\|^s$$
$$\le \frac{\left(2^{s-1}\right)}{1-2^{s-1}}\left\{\frac{18+2^{s+1}}{2}\right\}\lambda\,\|x\|^s$$
$$\le \frac{\left(2^{s-1}\right)\cdot 2}{2-2^{s}}\left\{\frac{18+2^{s+1}}{2}\right\}\lambda\,\|x\|^s$$
$$\le \frac{\left(2^{s-1}\right)^1\left(18+2^{s+1}\right)}{2-2^{s}}\lambda\,\|x\|^s$$

Case (ii): $L = \frac{1}{2^{s-1}}$ for $s>1$ if $i=1$.

$$\|f(2x,2x) - 8f(x,x) - A(x,x)\|$$
$$\le \frac{\left(\frac{1}{2^{s-1}}\right)^{1-1}}{1-\frac{1}{2^{s-1}}}\left\{\frac{18+2^{s+1}}{2}\right\}\lambda\,\|x\|^s$$
$$\le \frac{2^{s-1}}{2^{s-1}-1}\left\{\frac{18+2^{s+1}}{2}\right\}\lambda\,\|x\|^s$$
$$\le \frac{\left(2^{s-1}\right)\cdot 2}{2^{s}-2}\left\{\frac{18+2^{s+1}}{2}\right\}\lambda\,\|x\|^s$$
$$\le \frac{\left(2^{s-1}\right)\left(18+2^{s+1}\right)}{2^{s}-2}\lambda\,\|x\|^s$$

In similar manner we can prove the following cases $L = 2^{4s-1}$ for $s<1$ if $i=0$ and $L = \frac{1}{2^{4s-1}}$ for $s>1$ if $i=1$, and $L = 2^{4s-1}$ for $s<1$ if $i=0$ and $L = \frac{1}{2^{4s-1}}$ for $s>1$ if $i=1$ for (ii) and (iii) respectively. Hence the proof is complete.

Theorem 4.4: Let $f: U^2 \to V$ be a mapping for which there exist a function $\alpha: U^4 \to (0,\infty]$ with the condition

$$\lim_{n\to\infty}\frac{1}{\mu_i^{3n}}\alpha(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n w) = 0 \qquad \text{(IV.10)}$$

with $\mu_i = 2$ if $i=0$ and $\mu_i = \frac{1}{2}$ if $i=1$ such that the functional inequality

$$\|F(x,y,z,w)\| \le \alpha(x,y,z,w) \qquad \text{(IV.11)}$$

for all $x, y, z, w \in U$. If there exists $L = L(i) < 1$ such that the function

$$x \to \gamma(x) = \tfrac{1}{2}\beta(x),$$

has the property

$$\gamma(x) \le L \cdot \mu_i^3\,\gamma\left(\mu_i x\right). \qquad \text{(IV.12)}$$

Then there exists a unique 2-variable cubic mapping $C:U^2\to V$ satisfying (I.8) and

$$\|f(2x,2x)-2f(x,x)-C(x,x)\|\le\frac{L^{i-1}}{1-L}\gamma(x) \qquad \text{(IV.4)}$$

for all $x\in U$. The mapping $\beta(2^{kj}x)$ and $C(x,x)$ are defined in (III.10) and (III.33) respectively for all $x\in U$.

Proof: Consider the set

$$\Omega=\{p/p:U^2\to V, p(0)=0\}$$

and introduce the generalized metric on Ω,

$$\begin{aligned}d(g,h)&=d_\gamma(g,h)\\&=\inf\{K\in(0,\infty)/\|p(x)-q(x)\|\le K\gamma(x),x\in U\}\end{aligned}$$

It is easy to see that (Ω,d) is complete.

Define $T:\Omega^2\to\Omega$ by

$$Tp(x,x)=\frac{1}{\mu_i^3}p(\mu_i x,\mu_i x)$$

for all $x\in U$. Now for all $p,q\in\Omega$,

$$\begin{aligned}&d(g,h)\le K\\&\Rightarrow\|p(x,x)-q(x,x)\|\le K\gamma(x),\ x\in U,\\&\Rightarrow\left\|\frac{1}{\mu_i^3}p(\mu_i x,\mu_i x)-\frac{1}{\mu_i^3}q(\mu_i x,\mu_i x)\right\|\le\frac{1}{\mu_i^3}K\gamma(\mu_i x),\ x\in U,\\&\Rightarrow\left\|\frac{1}{\mu_i^3}p(\mu_i x,\mu_i x)-\frac{1}{\mu_i^3}q(\mu_i x,\mu_i x)\right\|\le LK\gamma(x),\ x\in U,\\&\Rightarrow\|T\,p(x,x)-T\,q(x,x)\|\le LK\gamma(x),\ x\in U,\\&\Rightarrow d_\gamma(Tp,Tq)\le LK.\end{aligned}$$

This gives $d(Tp,Tq)\le Ld(p,q)$, for all $p,q\in\Omega$, i.e., T is a strictly contractive mapping of Ω, with Lipschitz constant L.

From (III.36), we arrive

$$\left\|\frac{h(2x,2x)}{8}-h(x,x)\right\|\le\frac{\beta(x)}{8} \qquad \text{(IV.14)}$$

for all $x\in U$. Using (IV.14) for the case $i=0$ it reduces to

$$\left\|\frac{h(2x,2x)}{8}-h(x,x)\right\|\le L\gamma(x)$$

for all $x\in U$,

$$\text{i.e., } d_\gamma(h,Th)\le L\Rightarrow d(h,Th)\le L\le L^1<\infty$$

Again replacing $x=\frac{x}{2}$, in (IV.14), we get,

$$\left\|h(x,x)-8h\left(\frac{x}{2},\frac{x}{2}\right)\right\|\le\beta\left(\frac{x}{2}\right) \qquad \text{(IV.15)}$$

for all $x\in U$. Using (IV.14) for the case $i=1$ it reduces to

$$\left\|h(x,x)-8h\left(\frac{x}{2},\frac{x}{2}\right)\right\|\le\gamma(x)$$

for all $x\in U$,

$$\text{i.e., } d_\gamma(h,Th)\le 1\Rightarrow d(h,Th)\le 1\le L^0<\infty$$

Now from the fixed point alternative in both cases, it follows that there exists a fixed point C of T in Ω such that

$$C(x,x)=\lim_{n\to\infty}\frac{1}{\mu_i^{3n}}\left(f(\mu_i^{n+1}x,\mu_i^{n+1}x)-2f(\mu_i^n x,\mu_i^n x)\right) \qquad \text{(IV.16)}$$

for all $x\in U$, since $\lim_{n\to\infty}d(T^n h,C)=0$.

To prove $C:U^2\to V$ is cubic. Putting (x,y,z,w) by $\mu_i^n x,\mu_i^n y,\mu_i^n z,\mu_i^n w$ in (IV.11) and divide by μ_i^{3n}. It follows from (IV.1) that

$$\begin{aligned}\|C(x,y,z,w)\|&=\lim_{n\to\infty}\frac{\|F(\mu_i^n x,\mu_i^n y,\mu_i^n z,\mu_i^n w)\|}{\mu_i^{3n}}\\&\le\lim_{n\to\infty}\frac{\alpha(\mu_i^n x,\mu_i^n y,\mu_i^n z,\mu_i^n w)}{\mu_i^{3n}}=0\end{aligned}$$

for all $x,y,z,w\in U$, i.e., C satisfies (I.8).

According to the alternative fixed point, since C is the unique fixed point of T in the set $\Delta=\{p\in\Omega:d(f,p)<\infty\},C$ is the unique function such that

$$\|f(2x,2x)-8f(x,x)-A(x,x)\|\le K\gamma(x)$$

for all $x\in U$ and $K>0$. Again using the fixed point alternative, we obtain

$$d(f,C)\le\frac{1}{1-L}d(f,Tf)$$

this implies

$$d(f,C)\le\frac{L^{1-i}}{1-L}d(f,Tf)$$

which yields

$$\|f(2x,2x)-2f(x,x)-C(x,x)\|\le\frac{L^{1-i}}{1-L}\gamma(x)$$

this completes the proof of the theorem.

The following Corollary is an immediate consequence of Theorem 4.4 concerning the stability of (I.8).

Corollary 4.5: Let $F:U^2\to V$ be a mapping and there exists real numbers λ and s such that

$$\|F(x,y,z,w)\|\le\begin{cases}\lambda\{\|x\|^s+\|y\|^s+\|z\|^s+\|w\|^s\},\ s<1\ or\ s>1;\\ \lambda\|x\|^s\|y\|^s\|z\|^s\|w\|^s,\ s<\frac{3}{4}\ or\ s>\frac{3}{4};\\ \lambda\{\|x\|^s\|y\|^s\|z\|^s\|w\|^s+\{\|x\|^{4s}+\|y\|^{4s}+\|z\|^{4s}+\|w\|^{4s}\}\},\\ \qquad s<\frac{3}{4}\ or\ s>\frac{3}{4};\end{cases} \qquad \text{(IV.17)}$$

for all $x,y,z,w\in U$, then there exists a unique 2- variable cubic function $C:U^2\to V$ such that

$$\|f(2x,2x)-8f(x,x)-A(x,x)\|$$

$$\le \begin{cases} \dfrac{2^{s-1}\left(18+2^{s+1}\right)\lambda\|x\|^{s}}{\left|8-2^{s}\right|} \\ \dfrac{2^{4s-1}\left(4+2^{2s}\right)\lambda\|x\|^{4s}}{\left|8-2^{4s}\right|} \\ \dfrac{2^{4s-1}\left(22+2^{2s}+2\cdot 2^{4s}\right)\lambda\|x\|^{4s}}{\left|8-2^{4s}\right|} \end{cases} \tag{IV.18}$$

for all $x \in U$.

Proof: The proof of the corollary is similar tracing as that of corollary 4.3.

Theorem 4.6: Let $f: U^2 \to V$ be a mapping for which there exist a function $\alpha: U^4 \to (0,\infty]$ with (IV.1) and (IV.10) where $\mu_i = 2$ if $i=0$ and $\mu_i = \frac{1}{2}$ if $i=1$ such that the functional inequality

$$\|F(x,y,z,w)\| \le \alpha(x,y,z,w) \tag{IV.19}$$

for all $x,y,z,w \in U$. If there exists $L = L(i) < 1$ such that the function

$$x \to \gamma(x) = \tfrac{1}{2}\beta(x),$$

has the (IV.3) and (IV.12), then there exists a unique 2-variable additive mapping $A: U^2 \to V$ unique 2-variable cubic mapping $C: U^2 \to V$ satisfying (I.8) and

$$\| f(x,x) - A(x,x) - C(x,x)\| \le \frac{1}{3}\frac{L^{i-1}}{1-L}\gamma(x) \tag{IV.20}$$

for all $x \in U$. The mapping $\beta(2^{kj}x)$, $A(x,x)$ and $C(x,x)$ are defined in (III.5), (III.10) and (III.33) respectively for all $x \in U$.

Proof: By Theorems 4.1 and 4.4, there exists a unique 2-variable additive function $A_1: U^2 \to V$ and a unique 2-variable cubic function $C_1: U^2 \to V$ such that

$$\| f(2x,2x) - 8f(x,x) - A_1(x,x)\| \le \frac{L^{i-1}}{1-L}\gamma(x) \tag{IV.21}$$

and

$$\| f(2x,2x) - 2f(x,x) - C_1(x,x)\| \le \frac{L^{i-1}}{1-L}\gamma(x) \tag{IV.22}$$

for all $x \in U$. Now from (III.21) and (III.22), one can see that

$$\begin{aligned} &\left\| f(x,x) + \frac{1}{6}A_1(x,x) - \frac{1}{6}C_1(x,x)\right\| \\ &= \left\| \left\{ -\frac{f(2x,2x)}{6} + \frac{8f(x,x)}{6} + \frac{A_1(x,x)}{6} \right\} \right. \\ &\qquad \left. + \left\{ \frac{f(2x,2x)}{6} - \frac{2f(x,x)}{6} - \frac{C_1(x,x)}{6} \right\} \right\| \\ &\le \frac{1}{6}\{\| f(2x,2x) - 8f(x,x) - A_1(x,x)\| \\ &\qquad + \| f(2x,2x) - 8f(x,x) - A_1(x,x)\|\} \\ &\le \frac{1}{6}\left\{ \frac{L^{i-1}}{1-L}\gamma(x) + \frac{L^{i-1}}{1-L}\gamma(x) \right\} \end{aligned}$$

for all $x \in U$. Thus we obtain (IV.21) by defining $A_1(x,x) = \frac{-1}{6}A(x,x)$ and $C_1(x,x) = \frac{1}{6}C(x,x)$, $\beta(2^{kj}x)$, $A(x,x)$ and $C(x,x)$ are defined in (III.10), (III.5) and (III.33) for all $x \in U$.

The following corollary is the immediate consequence of Theorem 4.6, using Corollaries 4.3 and 4.5 concerning the stability of (I.8).

Corollary 4.7: Let $F: U^2 \to V$ be a mapping and there exists real numbers λ and s such that

$$\|F(x,y,z,w)\| \le \begin{cases} \lambda \\ \lambda\left\{\|x\|^{s} + \|y\|^{s} + \|z\|^{s} + \|w\|^{s}\right\}, \ s < 3 \ or \ s > 3; \\ \lambda\|x\|^{s}\|y\|^{s}\|z\|^{s}\|w\|^{s}, \ s < \dfrac{3}{4} \ or \ s > \dfrac{3}{4}; \\ \lambda\left\{\|x\|^{s}\|y\|^{s}\|z\|^{s}\|w\|^{s} + \left\{\|x\|^{4s} + \|y\|^{4s} + \|z\|^{4s} + \|w\|^{4s}\right\}\right\}, \\ \qquad s < \dfrac{3}{4} \ or \ s > \dfrac{3}{4}; \end{cases} \tag{IV.23}$$

for all $x,y,z,w \in U$, then there exists a unique 2-variable additive mapping $A: U^2 \to V$ unique 2-variable cubic mapping $C: U^2 \to V$ such that

$$\| f(x,x) - A(x,x) - C(x,x)\| \le \begin{cases} \dfrac{\left(18+2^{s+1}\right)}{3}\left(\dfrac{1}{\left|2-2^{s}\right|} + \dfrac{1}{7\left|8-2^{s}\right|}\right)\lambda\|x\|^{s} \\ \dfrac{\left(4+2^{2s}\right)}{3}\left(\dfrac{1}{\left|2-2^{4s}\right|} + \dfrac{1}{7\left|8-2^{4s}\right|}\right)\lambda\|x\|^{4s} \\ \dfrac{\left(22+2^{2s}+2\cdot 2^{4s}\right)}{3}\left(\dfrac{1}{\left|2-2^{4s}\right|} + \dfrac{1}{7\left|8-2^{2s}\right|}\right)\lambda\|x\|^{4s} \end{cases} \tag{III.24}$$

for all $x \in U$.

V. CONCLUSION

The (I.8) is stable in Banach spaces via direct and fixed point method in sprit of Hyers, Ulam, Rasssias.

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